Consequently,

$$\psi = \psi_r + i\psi_i = (1+ic)\psi_r = \tilde{c}\psi_r$$

Since the overall scale $\tilde{c}$ is irrelevant, we can ignore it, i.e., work with real eigenfunctions with no loss of generality.

This brings us to the end of our study of one-dimensional problems, except for the harmonic oscillator, which is the subject of Chapter 7.

Derivation of (6.1)

$|\Omega_1\rangle\, |\Omega_2\rangle\, |\Omega_3\rangle$

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \cdot \left\{ a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= a_1 \begin{bmatrix} \Omega_{11} \\ \Omega_{21} \\ \Omega_{33} \end{bmatrix} + a_2 \begin{bmatrix} \Omega_{12} \\ \Omega_{22} \\ \Omega_{32} \end{bmatrix} + a_3 \begin{bmatrix} \Omega_{13} \\ \Omega_{23} \\ \Omega_{33} \end{bmatrix} = \sum_i a_i |\Omega_i\rangle$$

$\langle \psi | \Omega | \psi \rangle$

$$= \left\{ a_1^* [1\ 0\ 0] + a_2^* [0\ 1\ 0] + a_3 [0\ 0\ 1] \right\} \left\{ a_1 \begin{bmatrix} \Omega_{11} \\ \Omega_{21} \\ \Omega_{31} \end{bmatrix} + a_2 \begin{bmatrix} \Omega_{12} \\ \Omega_{22} \\ \Omega_{32} \end{bmatrix} + a_3 \begin{bmatrix} \Omega_{13} \\ \Omega_{23} \\ \Omega_{33} \end{bmatrix} \right\}$$

$$= a_1^* a_1 \Omega_{11} + a_2^* a_2 \Omega_{22} + a_3^* a_3 \Omega_{33} = \sum_i a_i^* a_i \Omega_{ii}$$

$$\therefore \frac{d}{dt}\langle \psi | \Omega | \psi \rangle = \sum_i \dot{a}_i^* a_i \Omega_{ii} + a_i^* \dot{a}_i \Omega_{ii} + a_i^* a_i \dot{\Omega}_{ii}$$

$$= \langle \dot{\psi} | \Omega | \psi \rangle + \langle \psi | \Omega | \dot{\psi} \rangle + \langle \psi | \dot{\Omega} | \psi \rangle$$

//

P.196 Exercise 7.3.2

(7.3.21)より，$n=3$ および $n=4$ のエルミート多項式は

$$H_3(y) = -12\left(y - \frac{2}{3}y^3\right) \Rightarrow C_1 = 1, \quad C_3 = -\frac{2}{3}$$

$$H_4(y) = 12\left(1 - 4y^2 + \frac{4}{3}y^4\right) \Rightarrow C_0 = 1, \quad C_2 = -4, \quad C_4 = \frac{4}{3}$$

係数 C_n 間の制約は，

$$C_{n+2} = C_n \frac{(2n+1-2\varepsilon)}{(n+2)(n+1)} \quad \cdots \quad (7.3.15)$$

(1) $n=3$ のとき，

(7.3.15)より $\quad C_3 = C_1 \dfrac{3-2\varepsilon}{6}$

これに $C_1 = 1$, $C_3 = -2/3$ を代入すると，

$$-\frac{2}{3} = \frac{3-2\varepsilon}{6} \Rightarrow \varepsilon = \frac{7}{2} = \frac{1}{2} + 3$$

$$\therefore \quad E = \left(\frac{1}{2} + 3\right)\hbar\omega$$

(2) $n=4$ のとき，

$$C_2 = C_0 \frac{1-2\varepsilon}{2}, \qquad C_4 = C_2 \frac{5-2\varepsilon}{12}$$

これらに $C_0 = 1$, $C_2 = -4$, $C_4 = 4/3$ を代入して，

$$\begin{cases} -8-1 = -2\varepsilon \\ -4 = 5-2\varepsilon \end{cases} \Rightarrow \begin{cases} \varepsilon = \frac{1}{2} + 4 \\ \varepsilon = \frac{1}{2} + 4 \end{cases}$$

$$\therefore \quad E = \left(\frac{1}{2} + 4\right)\hbar\omega$$

P. 218 - Exercise 7.5.2

$$\begin{cases} a = \dfrac{1}{\sqrt{2}}\left(y + \dfrac{d}{dy}\right) & \cdots (7.5.3) \\ \langle n | y \rangle = \psi_n(y) = \left(\dfrac{m\omega}{\pi\hbar}\right)^{1/4} \dfrac{1}{(\sqrt{2})^n \sqrt{n!}} e^{-y^2/2} H_n(y) & \cdots (7.3.22) \end{cases}$$

$a|n\rangle = n^{1/2}|n-1\rangle$ を X 基底に投影すると.

$\leftarrow$ (7.4.21)

$$\langle y|a|n\rangle = n^{1/2}\langle y|n-1\rangle \quad --- ①$$

$$(①の左辺) = \langle y|a|y'\rangle\langle y'|n\rangle = a\,\delta(y-y')\langle y'|n\rangle$$

$$= a\langle y|n\rangle = a\psi_n(y)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{(\sqrt{2})^n\sqrt{n!}} \cdot \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) e^{-y^2/2} H_n(y)$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{(\sqrt{2})^n\sqrt{n!}} \frac{1}{\sqrt{2}} \left\{ y e^{-y^2/2} H_n(y) - y e^{-y^2/2} H_n(y) + e^{-y^2/2} \frac{dH_n(y)}{dy} \right\}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{(\sqrt{2})^n\sqrt{n!}} \frac{1}{\sqrt{2}} \cdot e^{-y^2/2} \cdot \frac{dH_n(y)}{dy}$$

$$(①の右辺) = \sqrt{n} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot \frac{1}{(\sqrt{2})^{n-1}\sqrt{(n-1)!}} e^{-y^2/2} H_{n-1}(y)$$

$$\therefore \quad \frac{dH_n(y)}{dy} = 2n\, H_{n-1}(y) \quad //$$

Exercise 7.5.3

$$\begin{cases} a + a^{+} = \sqrt{2}\, y \\ (a + a^{+}) | n \rangle = \sqrt{n}\, | n-1 \rangle + \sqrt{n+1}\, | n+1 \rangle \end{cases}$$

これに対して.

$$\psi_n(y) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n}\sqrt{n!}}\, e^{-y^2/2} H_n(y)$$ (7.3.22)を用いると.

$$\sqrt{2}\, y \cdot \frac{1}{\sqrt{2^n}\sqrt{n!}} H_n(y) = \sqrt{n}\, \frac{1}{\sqrt{2^{n-1}}\sqrt{n-1!}} H_{n-1}(y) + \sqrt{n+1}\, \frac{1}{\sqrt{2^{n+1}}\sqrt{n+1!}} H_{n+1}(y)$$

両辺に $\sqrt{2^{n-1}}\sqrt{n-1!}$ をかけると.

$$\frac{y}{\sqrt{n}} H_n(y) = \sqrt{n}\, H_{n-1}(y) + \frac{1}{2\sqrt{n}} H_{n+1}(y)$$

両辺に $2\sqrt{n}$ をかけて.

$$2y H_n(y) = 2n H_{n-1}(y) + H_{n+1}(y)$$

$$\therefore\ H_{n+1}(y) = 2y H_n(y) - 2n H_{n-1}(y)$$

P. 219. Exercise 7.5.4

(1) ボルツマンの公式

$$P(i) = e^{-\beta E(i)}/Z, \quad Z = \sum_i e^{-\beta E(i)}$$

$$\bar{E} = \sum_i \left\{ E(i) \cdot \frac{e^{-\beta E(i)}}{\sum_i e^{-\beta E(i)}} \right\} = \frac{1}{\sum_i e^{-\beta E(i)}} \sum_i E(i) e^{-\beta E(i)} \quad \cdots ①$$

ここで

$$-\frac{\partial}{\partial\beta} \ln Z = -\frac{\partial}{\partial\beta} \ln \left\{ \sum_i e^{-\beta E(i)} \right\} = \frac{\sum_i E(i) e^{-\beta E(i)}}{\sum_i e^{-\beta E(i)}} \quad \cdots ②$$

$$\therefore \quad ① = ② \quad //$$

(2)

$$Z_{cl} = \sum_i e^{-\beta E(i)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{-\beta\left(\frac{p^2}{2m} + \frac{1}{2} m w^2 x^2\right)\right\} dx\, dp$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{\beta}{2m}p^2} \cdot e^{-\frac{\beta m w^2}{2}x^2} \cdot dx \cdot dp$$

これらはガウス積分の形をしているので

$$= \sqrt{\frac{2m\pi}{\beta}} \cdot \sqrt{\frac{2\pi}{\beta m w^2}} = \frac{2\pi}{w\beta} \quad //$$

(← mやwによらない)

(3)

$$Z_{qu} = \sum_i e^{-\beta E(i)} = \sum_{i=0}^{n} e^{-\beta(\frac{1}{2}+i)\hbar w} = \sum_{i=0}^{n} e^{-\beta\hbar w/2} e^{-\beta i \hbar w}$$

$$= e^{-\beta\hbar w/2} \underbrace{\sum_{i=0}^{n} e^{-i\beta\hbar w}}_{f\text{とおく}}$$

(i: 自然数)

$$f - 1 = e^{-\beta\hbar w} \cdot f \iff (1 - e^{-\beta\hbar w}) f = 1$$

$$\iff f = \frac{1}{1 - e^{-\beta\hbar w}}$$

($n \to$ 大のとき)

$$\therefore \quad Z_{qu} = e^{-\beta\hbar w/2} \cdot \frac{1}{1 - e^{-\beta\hbar w}} \quad //$$

(ウラへ続く)

$$\bar{E}_{qu} = -\frac{\partial}{\partial\beta}\ln\left\{\frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}\right\} = -\frac{1-e^{-\beta\hbar\omega}}{e^{-\beta\hbar\omega/2}}\left\{\frac{-\frac{\hbar\omega}{2}e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}} - \frac{e^{-\beta\hbar\omega/2}}{(1-e^{-\beta\hbar\omega})^2}(\hbar\omega e^{-\beta\hbar\omega})\right\}$$

$$= \hbar\omega\left(\frac{1}{2}+\frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}\right) = \hbar\omega\left(\frac{1}{2}+\frac{1}{e^{\beta\hbar\omega}-1}\right)$$

(4)

$$\lim_{\beta\to 0}\bar{E}_{qu} \approx \hbar\omega\left(\frac{1}{2}+\frac{1}{(1+\beta\hbar\omega)-1}\right) = \hbar\omega\left(\frac{1}{2}+\frac{1}{\beta\hbar\omega}\right)$$

$$= \left(\frac{\hbar\omega}{2}+\frac{1}{\beta}\right) \approx kT = E_{cl}$$

これは $\hbar\omega \ll kT$ で成立.

(5)

$$C_{cl}(T) = \frac{1}{N_0}\frac{\partial}{\partial T}\{3N_0kT\} = 3k$$

$$C_{qu}(T) = \frac{1}{N_0}\frac{\partial}{\partial T}\left\{3N_0\hbar\omega\left(\frac{1}{2}+\frac{1}{e^{\frac{\hbar\omega}{kT}}-1}\right)\right\}$$

$$= 3\hbar\omega\cdot\frac{-\frac{\hbar\omega}{kT^2}e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}}-1)^2} = 3\hbar\omega\cdot\frac{\hbar\omega}{k^2T^2}\cdot k\cdot\frac{e^{\frac{\hbar\omega}{kT}}}{(e^{\frac{\hbar\omega}{kT}}-1)^2}$$

$\frac{\hbar\omega}{k} \equiv \theta_E$ とおけば

$$C_{qu}(T) = 3k\left(\frac{\theta_E}{T}\right)^2\frac{e^{\theta_E/T}}{(e^{\theta_E/T}-1)^2}$$

P.225, Fig. 8-3について　　　　　　　　　　　　　　　　($\hbar = 1.06\times10^{-27}$ erg·cm)

(1) 古典的な経路 $x = t$

$$v_{cl} = \frac{x - x'}{t - t'} \quad \Rightarrow \quad \mathcal{L}_{cl} = \frac{1}{2}m\left(\frac{x-x'}{t-t'}\right)^2$$

$$\therefore\ S_{cl} = \int_{t'}^{t} \mathcal{L}_{cl}\,dt'' = \frac{1}{2}m\left(\frac{x-x'}{t-t'}\right)^2(t-t') = \frac{1}{2}m\frac{(x-x')^2}{t-t'}$$

(2) 古典的に不可能な経路 $x = t^2$

$$v = \frac{dx}{dt''} = 2t'' \quad \Rightarrow \quad \mathcal{L} = \frac{1}{2}m(2t'')^2 = 2mt''^2$$

$$\therefore\ S = \int_{t'}^{t} \mathcal{L}\,dt' = \frac{2}{3}m(t'^3 - t^3)$$

コヒーレントである為の許容条件:

$$\delta S \equiv |S_{cl} - S| < \hbar\pi = \frac{h}{2} = 3.31\times10^{-27}\ [\mathrm{erg\cdot cm}],\quad \delta\theta < \pi\ [\mathrm{rad}]$$

(a) $m = 1\mathrm{g}$, $t - t' = 1\,\mathrm{sec}$ $(t=1,\ t'=0)$, $x - x' = 1\,\mathrm{cm}$ のとき.

$$S_{cl} = \frac{1}{2}\ \mathrm{erg\cdot cm} \qquad S = \frac{2}{3}\ \mathrm{erg\cdot cm}$$

$$\Rightarrow\ \delta S = \frac{1}{6}\ \mathrm{erg\cdot cm} \quad \Rightarrow \text{位相のズレは}\ \frac{\delta S}{\hbar} = 1.57\times10^{26}\ [\mathrm{rad}]$$

(b) $m = 10^{-27}\mathrm{g}$ (←電子), $t - t' = 1\,\mathrm{sec}$, $x - x' = 1\,\mathrm{cm}$ のとき.

$$S_{cl} = \frac{1}{2}\times10^{-27}\ \mathrm{erg\cdot cm},\quad S = \frac{2}{3}\times10^{-27}\ \mathrm{erg\cdot cm}$$

$$\delta S = \frac{1}{6}\times10^{-27}\ \mathrm{erg\cdot cm} \quad \Rightarrow \text{位相のズレは}\ \frac{\delta S}{\hbar} \simeq \frac{1}{6}\ [\mathrm{rad}]$$

P.226 の A'の決め方について。　$\delta t \equiv t - t'$ とおく。

$$\lim_{\delta t\to 0} U = \lim_{\delta t \to 0} A' \exp\left\{\frac{im(x-x')^2}{2\hbar\,\delta t}\right\} = \delta(x-x') \quad \cdots ①$$

一方、

$$\lim_{\Delta^2\to 0} \frac{1}{\sqrt{\pi\Delta^2}} \exp\left\{-\frac{(x-x')^2}{\Delta^2}\right\} = \delta(x-x') \quad \cdots ②$$

② で $\Delta^2 \to \dfrac{2\hbar i\,\delta t}{m}$ と変数変換すると、

$$\lim_{\delta t\to 0} \sqrt{\frac{m}{2\pi\hbar i\,\delta t}} \exp\left\{\frac{m(x-x')^2}{2\hbar i\,\delta t}\right\} = \delta(x-x') \quad \cdots ③$$

① と ③ を比較すると、

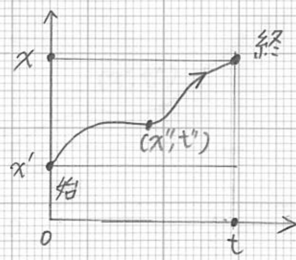
$$A' = \left(\frac{m}{2\pi\hbar i\,\delta t}\right)^{\frac{1}{2}} = \left(\frac{m}{2\pi\hbar i\,(t-t')}\right)^{\frac{1}{2}}$$

これを (8.3.3) に代入すると プロパゲーター ($t'=0$) は、

$$U(x,t;x',0) = \left(\frac{m}{2\pi\hbar i t}\right)^{\frac{1}{2}} \exp\left\{\frac{im(x-x')^2}{2\hbar t}\right\} \quad (8.3.4)$$

これは (5.1.10) と同じです。

p. 234 Exercise 8.6.1



一定の大きさの作用力を与えるポテンシャル $V(x) = -fx$
中でのプロパゲーターを求める.

$$F = \frac{dV}{dx} = -f = m\ddot{x}_f \Rightarrow x_f = -\frac{f}{2m}t''^2$$

$$x_{cl}(t'') = \overbrace{x_0 + v_0 t''}^{\text{等速直線運動}} + \overbrace{x_f}^{\text{加速運動}} = x_0 + v_0 t'' + \frac{f}{2m}t''^2$$ とひとまずおく.

$$\begin{cases} x_{cl}(0) = x' \iff x_0 = x' \\ x_{cl}(t) = x \iff x' + v_0 t + \frac{f}{2m}t^2 = x \iff v_0 = \frac{x-x'}{t} - \frac{ft}{2m} \end{cases}$$

$$\therefore x_{cl}(t'') = x' + \left(\frac{x-x'}{t} - \frac{ft}{2m}\right)t'' + \frac{f}{2m}t''^2$$

$$\mathcal{L}_{cl} = \frac{1}{2}m\dot{x}_{cl}^2 - V(x'') = \frac{1}{2}\cdot m\cdot\left\{\left(\frac{x-x'}{t} - \frac{ft}{2m}\right) + \frac{ft''}{m}\right\}^2 + f x_{cl}(t'')$$

$$S_{cl} = \int_0^t \mathcal{L}_{cl}\,dt'' = \int_0^t \frac{1}{2}m\left\{\left(\frac{x-x'}{t} - \frac{ft}{2m}\right)^2 + \left(\frac{x-x'}{t} - \frac{ft}{2m}\right)\frac{2ft''}{m} + \frac{f^2t''^2}{m^2}\right\}$$

$$+ f\left\{x' + \left(\frac{x-x'}{t} - \frac{ft}{2m}\right)t'' + \frac{f}{2m}t''^2\right\}\cdot dt''$$

$$= \frac{1}{2}m\frac{(x-x')^2}{t} - \frac{1}{2}ft(x-x') + \frac{f^2t^3}{8m} + \frac{1}{2}ft(x-x') - \frac{f^2t^3}{4m}$$

$$+ \frac{f^2t^3}{6m} + fx't + \frac{1}{2}f(x-x') - \frac{f^2t^3}{4m} + \frac{f^2t^3}{6m}$$

$$= \frac{m(x-x')^2}{2t} + \frac{1}{2}ft(x+x') - \frac{f^2t^3}{24m}$$

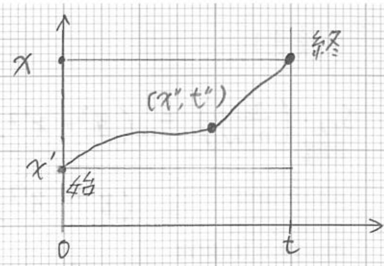
$$\therefore U(x,t;x') = A'\exp\left(\frac{iS_{cl}}{\hbar}\right)$$

$$= \left(\frac{m}{2\pi\hbar i t}\right)^{\frac{1}{2}}\exp\left\{\frac{i}{\hbar}\left[\frac{m(x-x')^2}{2t} + \frac{1}{2}ft(x+x') - \frac{f^2t^3}{24m}\right]\right\}$$

(5.4.31)に等しくなっている.

Exercise 8.6.2 (P.234)

一次元調和振動子のプロパゲーターを求める.



古典的な軌道は.

$$x_{cl}(t'') = A\cos wt'' + B\sin wt''$$

$t''=0$ で $x_{cl} = x'$ $\Rightarrow$ $A = x'$

$t''=t$ で $x_{cl} = x$ $\Rightarrow$ $x'\cos wt + B\sin wt = x$

$$\Leftrightarrow \quad B = \frac{x - x'\cos wt}{\sin wt}$$

$$\therefore \quad x_{cl}(t'') = x'\cos wt'' + B\sin wt''$$

$$\dot{x}_{cl}(t'') = -wx'\sin wt'' + Bw\cos wt''$$

古典的なラグランジュアンは.

$$\mathcal{L}_{cl} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}mw^2x^2$$

$$\Leftrightarrow \frac{2}{m}\cdot\mathcal{L}_{cl} = \dot{x}^2 - w^2x^2$$

$$= w^2x'^2\sin^2wt'' - 2w^2x'B\sin wt''\cos wt'' + B^2w^2\cos^2wt''$$

$$-w^2(x'^2\cos^2wt'' + 2x'B\cos wt''\sin wt'' + B^2\sin^2wt'')$$

$$\Leftrightarrow \frac{2}{mw^2}\mathcal{L}_{cl} = x'^2(\sin^2wt'' - \cos^2wt'') - 4x'B\cos wt''\sin wt''$$

$$+ B^2(\cos^2wt'' - \sin^2wt'')$$

$$= (B^2 - x'^2)(\cos^2wt'' - \sin^2wt'') - 4x'B\cos wt''\sin wt''$$

$$= (B^2 - x'^2)\cos 2wt'' - 2Bx'\sin 2wt''$$

$$\therefore \quad S_{cl}\cdot\left(\frac{2}{mw^2}\right) = \int_0^t \mathcal{L}_{cl}\cdot\left(\frac{2}{mw^2}\right)dt''$$

$$= \left[\frac{B^2 - x'^2}{2w}\sin 2wt'' + \frac{Bx'}{w}\cos 2wt''\right]_0^t$$

$$= \frac{B^2 - x'^2}{2w}\sin 2wt + \frac{Bx'}{w}(\cos 2wt - 1)$$

(ウラ側へ続く.)

(続き)

$$S_{cl}\cdot\left(\frac{2}{m\omega}\right) = (B^2 - x'^2)\sin\omega t\cos\omega t + Bx'(1 - 2\sin^2\omega t - 1)$$

$$= (B^2 - x'^2)\sin\omega t\cos\omega t - 2Bx'\sin^2\omega t$$

$$= \left(\frac{x^2 - 2xx'\cos\omega t + x'^2\cos^2\omega t - x'^2\sin^2\omega t}{\sin^2\omega t}\right)\sin\omega t\cos\omega t$$

$$- \frac{2(xx' - x'^2\cos\omega t)}{\sin\omega t}\cdot\sin^2\omega t$$

$$\Leftrightarrow S_{cl}\left(\frac{2\sin\omega t}{m\omega}\right) = (x^2 + x'^2\cos^2\omega t - x'^2\sin^2\omega t)\cos\omega t$$

$$- 2xx'\cos^2\omega t - 2xx'\sin^2\omega t + 2x'^2\sin^2\omega t\cos\omega t$$

$$= (x^2 + x'^2\cos^2\omega t - x'^2\sin^2\omega t + 2x'^2\sin^2\omega t)\cos\omega t - 2xx'$$

$$= (x^2 + x'^2)\cos\omega t - 2xx'$$

$$\therefore S_{cl} = \frac{m\omega}{2\sin\omega t}\left[(x^2 + x'^2)\cos\omega t - 2xx'\right]$$

$$\therefore U(x, t; x') = A(t)\exp\left\{\frac{im\omega}{2\hbar\sin\omega t}\left[(x^2 + x'^2)\cos\omega t - 2xx'\right]\right\}$$

(P.196 の (7.3.28) において

$$A(t) = \left(\frac{m\omega}{2\pi\hbar i\sin\omega t}\right)^{\frac{1}{2}}$$ となっている。)

Exercise 10.1.1

(1) $[\Omega_1^{(1)} \otimes I^{(2)}, I^{(1)} \otimes \Lambda_2^{(2)}] \cdot (|x_1\rangle \otimes |x_2\rangle)$

$= (\Omega_1^{(1)} \otimes I^{(2)})(I^{(1)} \otimes \Lambda_2^{(2)}) |x_1\rangle \otimes |x_2\rangle - (I^{(1)} \otimes \Lambda_2^{(2)})(\Omega_1^{(1)} \otimes I^{(2)}) |x_1\rangle \otimes |x_2\rangle$

$= (\Omega_1^{(1)} \otimes I^{(2)}) |x_1\rangle \otimes |\Lambda_2^{(2)} x_2\rangle - (I^{(1)} \otimes \Lambda_2^{(2)}) |\Omega_1^{(1)} x_1\rangle \otimes |x_2\rangle$

$= |\Omega_1^{(1)} x_1\rangle \otimes |\Lambda_2^{(2)} x_2\rangle - |\Omega_1^{(1)} x_1\rangle \otimes |\Lambda_2^{(2)} x_2\rangle = 0$

(2)

$(\Omega_1^{(1)} \otimes \Gamma_2^{(2)})(\Theta_1^{(1)} \otimes \Lambda_2^{(2)}) \cdot (|x_1\rangle \otimes |x_2\rangle)$

$= (\Omega_1^{(1)} \otimes \Gamma_2^{(2)}) \cdot |\Theta_1^{(1)} x_1\rangle \otimes |\Lambda_2^{(2)} x_2\rangle = |\Omega_1^{(1)} \Theta_1^{(1)} x_1\rangle \otimes |\Gamma_2^{(2)} \Lambda_2^{(2)} x_2\rangle$

$= \{(\Omega\Theta)_1^{(1)} \otimes (\Gamma\Lambda)_2^{(2)}\} |x_1\rangle \otimes |x_2\rangle$

(3) When

$[\Omega_1^{(1)}, \Lambda_1^{(1)}] |x_1\rangle = (\Omega_1^{(1)} \Lambda_1^{(1)} - \Lambda_1^{(1)} \Omega_1^{(1)}) |x_1\rangle = \Gamma_1^{(1)} |x_1\rangle$

$[\Omega_1^{(1)\otimes(2)}, \Lambda_1^{(1)\otimes(2)}] (|x_1\rangle \otimes |x_2\rangle) = (\Omega_1^{(1)\otimes(2)} \Lambda_1^{(1)\otimes(2)} - \Lambda_1^{(1)\otimes(2)} \Omega_1^{(1)\otimes(2)}) |x_1\rangle \otimes |x_2\rangle$

$= |(\Omega_1^{(1)} \Lambda_1^{(1)} - \Lambda_1^{(1)} \Omega_1^{(1)}) x_1\rangle \otimes |I_2 x_2\rangle = |\Gamma^{(1)} x_1\rangle \otimes |I_2 x_2\rangle$

$= \Gamma^{(1)} \otimes I^{(2)} |x_1\rangle \otimes |x_2\rangle$

(4)

直積は線形演算だから、

$(\Omega_1^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)})^2 = \Omega_1^{(1)\otimes(2)} \cdot \Omega_1^{(1)\otimes(2)} + \Omega_1^{(1)\otimes(2)} \Omega_2^{(1)\otimes(2)} + \Omega_2^{(1)\otimes(2)} \Omega_1^{(1)\otimes(2)}$
$+ \Omega_2^{(1)\otimes(2)} \Omega_2^{(1)\otimes(2)}$

$= (\Omega_1^{(1)})^2 \otimes I^{(2)} + I^{(1)} \otimes (\Omega_2^{(2)})^2 + 2 \cdot \Omega_1^{(1)} \otimes \Omega_2^{(2)}$

Exercise 10.1.2

(1) $\sigma_1^{(1)\otimes(2)} = \sigma_1^{(1)} \otimes I^{(2)}$ マトリックスエレメントは.

$$\langle + | \otimes \langle + | \sigma_1^{(1)} \otimes I^{(2)} | + \rangle \otimes | + \rangle = \langle + | \sigma_1^{(1)} | + \rangle \langle + | I^{(2)} | + \rangle = a$$

$$\langle + | \otimes \langle - | \sigma_1^{(1)} \otimes I^{(2)} | + \rangle \otimes | + \rangle = \langle + | \sigma_1^{(1)} | + \rangle \langle - | I^{(2)} | + \rangle = 0$$

$$\langle - | \otimes \langle + | \sigma_1^{(1)} \otimes I^{(2)} | + \rangle \otimes | + \rangle = \langle - | \sigma_1^{(1)} | + \rangle \langle + | I^{(2)} | + \rangle = c$$

$$\langle - | \otimes \langle + | \sigma_1^{(1)} \otimes I^{(2)} | - \rangle \otimes | + \rangle = \langle - | \sigma_1^{(1)} | - \rangle \langle + | I^{(2)} | + \rangle = d$$

$\vdots$

(2) $\sigma_2^{(1)\otimes(2)} = I^{(1)} \otimes \sigma_2^{(2)}$ マトリックスエレメントは.

$$\langle + | \otimes \langle + | I^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | + \rangle = \langle + | I^{(1)} | + \rangle \langle + | \sigma_2^{(2)} | + \rangle = e$$

$$\langle + | \otimes \langle - | I^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | + \rangle = \langle + | I^{(1)} | + \rangle \langle - | \sigma_2^{(2)} | + \rangle = g$$

$$\langle + | \otimes \langle + | I^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | - \rangle = \langle + | I^{(1)} | + \rangle \langle + | \sigma_2^{(2)} | - \rangle = f$$

$$\langle + | \otimes \langle - | I^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | - \rangle = \langle + | I^{(1)} | + \rangle \langle - | \sigma_2^{(2)} | - \rangle = h$$

$\vdots$

(3) Way 1

$$(\sigma_1 \sigma_2)^{(1)\otimes(2)} | x_1 \rangle \otimes | x_2 \rangle = \sigma_1^{(1)} \otimes \sigma_2^{(2)} | x_1 \rangle \otimes | x_2 \rangle = (\sigma_1^{(1)} \otimes I^{(2)})(I^{(1)} \otimes \sigma_2^{(2)}) | x_1 \rangle \otimes |$$

$$= \sigma_1^{(1)\otimes(2)} \sigma_2^{(1)\otimes(2)} | x_1 \rangle \otimes | x_2 \rangle$$

$$\because \begin{cases} \sigma_1^{(1)} \otimes \sigma_2^{(2)} | x_1 \rangle \otimes | x_2 \rangle = | \sigma_1^{(1)} x_1 \rangle \otimes | \sigma_2^{(2)} x_2 \rangle \\ (\sigma_1^{(1)} \otimes I^{(2)})(I^{(1)} \otimes \sigma_2^{(2)}) | x_1 \rangle \otimes | x_2 \rangle = | \sigma_1^{(1)} x_1 \rangle \otimes | \sigma_2^{(2)} x_2 \rangle \end{cases}$$

$$\therefore (\sigma_1 \sigma_2)^{(1)\otimes(2)} = \sigma_1^{(1)\otimes(2)} \sigma_2^{(1)\otimes(2)}$$

Way 2

$\sigma_1^{(1)} \otimes \sigma_2^{(2)}$ のマトリックスエレメントは.

$$\langle + | \otimes \langle + | \sigma_1^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | + \rangle = \langle + | \sigma_1^{(1)} | + \rangle \langle + | \sigma_2^{(2)} | + \rangle = ae$$

$$\langle + | \otimes \langle - | \sigma_1^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | + \rangle = \langle + | \sigma_1^{(1)} | + \rangle \langle - | \sigma_2^{(2)} | + \rangle = ag$$

$$\langle + | \otimes \langle + | \sigma_1^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | - \rangle = \langle + | \sigma_1^{(1)} | + \rangle \langle + | \sigma_2^{(2)} | - \rangle = af$$

$$\langle + | \otimes \langle - | \sigma_1^{(1)} \otimes \sigma_2^{(2)} | + \rangle \otimes | - \rangle = \langle + | \sigma_1^{(1)} | + \rangle \langle - | \sigma_2^{(2)} | - \rangle = ah$$

$\vdots$

Exercise 11.2.1

$$\langle \psi_\varepsilon | P | \psi_\varepsilon \rangle = \int_{-\infty}^{\infty} \langle \psi_\varepsilon | x \rangle \langle x | P | x \rangle \langle x | \psi_\varepsilon \rangle \, dx$$

ここで、$T^\dagger(\varepsilon) T(\varepsilon) = \mathbb{I} \iff T^\dagger(\varepsilon) = [T(\varepsilon)]^{-1} = T(-\varepsilon)$ $\because$ T : ユニタリ

および

$T(\varepsilon)|\psi\rangle = |\psi_\varepsilon\rangle$ (11.2.3) $\iff \langle \psi_\varepsilon | = \langle \psi | T^\dagger(\varepsilon)$ を用いると、

$$\langle \psi_\varepsilon | x \rangle = \langle \psi | T^\dagger(\varepsilon) | x \rangle = \langle \psi | T(-\varepsilon) | x \rangle$$

$$= \langle \psi | e^{-i\varepsilon g(x)/\hbar} | x - \varepsilon \rangle = e^{-i\varepsilon g(x)/\hbar} \psi(x-\varepsilon)^*$$

$$\left(\because \ T(\varepsilon)|x\rangle = e^{i\varepsilon g(x)/\hbar} |x+\varepsilon\rangle \ \cdots (11.2.10) \right)$$

$$\langle x | \psi_\varepsilon \rangle = \langle x | T(\varepsilon) | \psi \rangle = \langle x | T^\dagger(-\varepsilon) | \psi \rangle$$

$$= \langle x | T^\dagger(-\varepsilon) | \psi \rangle = \langle x - \varepsilon | e^{-i(-\varepsilon) g(x)/\hbar} | \psi \rangle$$

$$= e^{i\varepsilon g(x)/\hbar} \langle x - \varepsilon | \psi \rangle = e^{i\varepsilon g(x)/\hbar} \psi(x-\varepsilon)$$

$$\left(\because \ \langle x | T^\dagger(\varepsilon) = \langle x + \varepsilon | e^{-i\varepsilon g(x)/\hbar} \right)$$

$$\therefore \ \langle \psi_\varepsilon | P | \psi_\varepsilon \rangle = \int_{-\infty}^{\infty} e^{-i\varepsilon g(x)/\hbar} \psi^*(x-\varepsilon) \left(-i\hbar \frac{d}{dx} \right) e^{i\varepsilon g(x)/\hbar} \psi(x-\varepsilon) \, dx$$

$$= \int_{-\infty}^{\infty} e^{-i\varepsilon g(x)/\hbar} \psi^*(x-\varepsilon) \left\{ -i\hbar \cdot \frac{i\varepsilon}{\hbar} \cdot e^{i\varepsilon g(x)/\hbar} \cdot \frac{dg(x)}{dx} \cdot \psi(x-\varepsilon) + e^{i\varepsilon g(x)/\hbar} \left(-i\hbar \frac{d}{dx} \right) \psi(x-\varepsilon) \right\} dx$$

$$= \int_{-\infty}^{\infty} \psi^*(x-\varepsilon) \left(-i\hbar \frac{d}{dx} \right) \psi(x-\varepsilon) \, dx$$

$$+ \int_{-\infty}^{\infty} \psi^*(x-\varepsilon) \cdot \varepsilon \cdot \frac{dg(x)}{dx} \psi(x-\varepsilon) \, dx$$

$$= \langle P \rangle + \varepsilon \left\langle \frac{dg(x)}{dx} \right\rangle$$

Exercise 11.2.2

$$T^+(\varepsilon)T(\varepsilon) = \left(I + \frac{i\varepsilon}{\hbar}G^+\right)\left(I - \frac{i\varepsilon}{\hbar}G\right)$$

$$= I - \frac{i\varepsilon}{\hbar}(G - G^+) + \underbrace{\frac{\varepsilon^2}{\hbar^2}G^+G}_{\simeq O(\varepsilon^2)} = I \quad \therefore \quad G = G^+$$

Eq. 12.2.25　P, L の交換関係の導出

$$\left(I+\frac{i}{\hbar}\varepsilon_z L_z\right)\left[I+\frac{i}{\hbar}(\varepsilon_x P_x+\varepsilon_y P_y)\right]\left(I-\frac{i}{\hbar}\varepsilon_z L_z\right)\left[I-\frac{i}{\hbar}(\varepsilon_x P_x+\varepsilon_y P_y)\right]$$

$$= I \cancel{-\frac{i}{\hbar}(\varepsilon_x P_x+\varepsilon_y P_y)} \cancel{-\frac{i}{\hbar}\varepsilon_z L_z} \cancel{-\frac{1}{\hbar^2}\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)}$$

$$\cancel{+\frac{i}{\hbar}(\varepsilon_x P_x+\varepsilon_y P_y)} \cancel{+\frac{1}{\hbar^2}(\varepsilon_x P_x+\varepsilon_y P_y)^2} + \frac{1}{\hbar^2}(\varepsilon_x P_x+\varepsilon_y P_y)\varepsilon_z L_z$$

$$\cancel{-\frac{i}{\hbar^3}(\varepsilon_x P_x+\varepsilon_y P_y)\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)}$$

$$\cancel{+\frac{i}{\hbar}\varepsilon_z L_z} \cancel{+\frac{1}{\hbar^2}\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)} \cancel{+\frac{1}{\hbar^2}\varepsilon_z^2 L_z^2} \cancel{-\frac{i}{\hbar^3}\varepsilon_z^2 L_z^2(\varepsilon_x P_x+\varepsilon_y P_y)}$$

$$-\frac{1}{\hbar^2}\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y) \cancel{+\frac{i}{\hbar^3}\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)^2}$$

$$\cancel{+\frac{1}{\hbar^3}\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)\varepsilon_z L_z} \cancel{+\frac{1}{\hbar^4}\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)\varepsilon_z L_z(\varepsilon_x P_x+\varepsilon_y P_y)}$$

上式で、$O(\varepsilon^3)$, $O(\varepsilon^4)$, ε_x^2, ε_y^2, ε_z^2, $\varepsilon_x\varepsilon_y$ を無視し、かつ $[P_x, P_y]=0$ (12.1.8) を用いると、

$$= I+\frac{1}{\hbar^2}\left\{\varepsilon_z\varepsilon_x(P_x L_z-L_z P_x)+\varepsilon_y\varepsilon_z(P_y L_z-L_z P_y)\right\}$$

$$\left.\begin{aligned} &= I+\frac{1}{\hbar^2}\varepsilon_z\varepsilon_x[P_x, L_z]+\frac{1}{\hbar^2}\varepsilon_y\varepsilon_z[P_y, L_z] \\ &= I+\frac{i}{\hbar}\varepsilon_y\varepsilon_z P_x-\frac{i}{\hbar}\varepsilon_x\varepsilon_z P_y \end{aligned}\right\} \cdots (*)$$

上の(*)の両辺を比較して、

$$[P_x, L_z] = -i\hbar P_y$$

$$[P_y, L_z] = i\hbar P_x$$

Exercise 12.2.3

$$x = r\cos\phi, \quad y = r\sin\phi \iff \tan\phi = \frac{y}{x} \iff \phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(X)$$

ここで、$X \equiv \dfrac{y}{x}$　$\dfrac{d}{dX}\tan^{-1}(X) = \dfrac{1}{1+X^2}$ を用いて.

$$\frac{\partial\phi}{\partial x} = \frac{1}{1+X^2}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2} = -\frac{\sin\phi}{r}$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{1+X^2}\cdot\frac{1}{x} = \frac{x}{r^2} = \frac{\cos\phi}{r}$$

$$L_z \longrightarrow x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right)$$ (12.2.10) について.

$$L_z \longrightarrow r\cos\phi\left(-i\hbar\frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial y}\right) - r\sin\phi\left(-i\hbar\frac{\partial}{\partial\phi}\cdot\frac{\partial\phi}{\partial x}\right)$$

$$= r\cos\phi\left(-i\hbar\frac{\partial}{\partial\phi}\cdot\frac{\cos\phi}{r}\right) - r\sin\phi\left(-i\hbar\frac{\partial}{\partial\phi}\cdot -\frac{\sin\phi}{r}\right)$$

$$= r(\cos^2\phi + \sin^2\phi)\left(-i\hbar\cdot\frac{1}{r}\cdot\frac{\partial}{\partial\phi}\right)$$

$$= -i\hbar\frac{\partial}{\partial\phi}$$

//

Exercise 12.3.1

(12.3.5) $\Longleftrightarrow$

$$-i\hbar\int_0^\infty\left\{\int_0^{2\pi}\psi_1{}^*\frac{\partial\psi_2}{\partial\phi}d\phi\right\}\rho d\rho = \left[-i\hbar\int_0^\infty\left\{\int_0^{2\pi}\psi_2{}^*\frac{\partial\psi_1}{\partial\phi}d\phi\right\}\rho d\rho\right]^*$$

$\Longleftrightarrow$

$$-i\hbar\int_0^{2\pi}\psi_1{}^*\frac{\partial\psi_2}{\partial\phi}d\phi = \left[-i\hbar\int_0^{2\pi}\psi_2{}^*\frac{\partial\psi_1}{\partial\phi}d\phi\right]^* \quad \cdots\cdots (*)$$

ここで.

$$\int_0^{2\pi}\psi_1{}^*\frac{\partial\psi_2}{\partial\phi}d\phi = \left[\psi_1{}^*\psi_2\right]_0^{2\pi} - \int_0^{2\pi}\frac{\partial\psi_1{}^*}{\partial\phi}\psi_2\,d\phi$$

$$\int_0^{2\pi}\psi_2{}^*\frac{\partial\psi_1}{\partial\phi}d\phi = \left[\psi_1\psi_2{}^*\right]_0^{2\pi} - \int_0^{2\pi}\psi_1\frac{\partial\psi_2{}^*}{\partial\phi}d\phi$$

これらを(*)に代入すると.

$$-\left[\psi_1{}^*\psi_2\right]_0^{2\pi} + \int_0^{2\pi}\frac{\partial\psi_1{}^*}{\partial\phi}\psi_2\,d\phi = \left[\psi_1{}^*\psi_2\right]_0^{2\pi} - \int_0^{2\pi}\psi_1{}^*\frac{\partial\psi_2}{\partial\phi}d\phi$$

$$\Longleftrightarrow\quad 2\left[\psi_1{}^*\psi_2\right]_0^{2\pi} - \int_0^{2\pi}\left(\frac{\partial\psi_1{}^*}{\partial\phi}\psi_2 + \psi_1{}^*\frac{\partial\psi_2}{\partial\phi}\right)d\phi = 0$$

$$\Longleftrightarrow\quad 2\left[\psi_1{}^*\psi_2\right]_0^{2\pi} - \left[\psi_1{}^*\psi_2\right]_0^{2\pi} = \left[\psi_1{}^*\psi_2\right]_0^{2\pi} = 0.$$

いかなるψにおいてもこれが満たされる為には、$\psi(\rho, 0) = \psi(\rho, 2\pi)$ でなくてはならない。

$\Rightarrow$ これは (12.3.3) $\psi_{l_z}(\rho,\phi) = R(\rho)\,e^{il_z\phi/\hbar}$ において.

少なくとも l_z は実数であり、より具体的には

$l_z = m\hbar$ ($m = 0, \pm1, \pm2, \cdots$) であることを意味している

xercise 12.3.3

$$\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}$$

$$\cos^2\phi = \frac{1}{4}\left(e^{2i\phi}+2+e^{-2i\phi}\right)$$

$$= \frac{\sqrt{2\pi}}{4}\left(\frac{2}{\sqrt{2\pi}}+\frac{1}{\sqrt{2\pi}}e^{i2\phi}+\frac{1}{\sqrt{2\pi}}e^{-i2\phi}\right)$$

$$\therefore\ \psi(\rho,\phi) = A'\cdot e^{-\rho^2/2\Delta^2}\cdot\left\{2\Phi_0(\phi)+\Phi_2(\phi)+\Phi_{-2}(\phi)\right\} \quad \left(A' \equiv \frac{\sqrt{2\pi}}{4}A\right)$$

ここで. P.118 (4.2.1) より

$$P(\Phi_i) = \frac{|\langle\Phi_i|\psi\rangle|^2}{\sum_j|\langle\Phi_j|\psi\rangle|^2} \quad (i=0,\ 2,\ -2)$$

(12.3.10) の直交条件を用いて

$$P(\Phi_0) = \frac{|\Phi_0\cdot\psi(\rho,\phi)|^2}{A'^2e^{-\rho^2/\Delta^2}(2^2+1^2+1^2)} = \frac{A'^2\cdot e^{-\rho^2/\Delta^2}\cdot 4}{A'^2\cdot e^{-\rho^2/\Delta^2}\cdot 6} = \frac{2}{3}$$

$$P(\Phi_2) = \frac{|\Phi_2\cdot\psi(\rho,\phi)|^2}{\text{〃}} = \frac{1}{6}$$

$$P(\Phi_{-2}) = \frac{|\Phi_{-2}\cdot\psi(\rho,\phi)|^2}{\text{〃}} = \frac{1}{6}$$

//

Exercise 12.3.4

$$\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}, \quad \sin\phi = \frac{e^{i\phi}-e^{-i\phi}}{2i}$$

$$\psi(\rho,\phi) = A\, e^{-\rho^2/2\Delta^2}\left(\frac{\rho}{\Delta}\cdot\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\right)$$

$$= A\, e^{-\rho^2/2\Delta^2}\cdot\frac{\sqrt{2\pi}}{2}\left\{\frac{\rho}{\Delta}\frac{1}{\sqrt{2\pi}}(e^{i\phi}+e^{-i\phi}) - \frac{i}{\sqrt{2\pi}}(e^{i\phi}-e^{-i\phi})\right\}$$

$$= A'\left\{\left(\frac{\rho}{\Delta}-i\right)\Phi_1(\phi) + \left(\frac{\rho}{\Delta}+i\right)\Phi_{-1}(\phi)\right\}$$

$$P(\Phi_1) = \frac{|\langle\Phi_1|\psi\rangle|^2}{\sum_j|\langle\Phi_j|\psi\rangle|^2} = \frac{\left|\frac{\rho}{\Delta}-i\right|^2}{\left|\frac{\rho}{\Delta}-i\right|^2+\left|\frac{\rho}{\Delta}+i\right|^2} = \frac{1}{2}$$

$$P(\Phi_2) = \frac{|\langle\Phi_2|\psi\rangle|^2}{\sum_j|\langle\Phi_j|\psi\rangle|^2} = \frac{\left|\frac{\rho}{\Delta}+i\right|^2}{\text{〃}} = \frac{1}{2}$$

$$\therefore\ P(l_z=\hbar) = P(l_z=-\hbar) = \frac{1}{2}$$

Exercise 12.3.5

(12.3.13) から

$$\left\{ \frac{-\hbar^2}{2\mu}\left(-\frac{m^2}{\rho^2}\right) + V(\rho) \right\} R_{Em}(\rho) = E R_{Em}(\rho)$$

$$\Longrightarrow \quad \frac{\hbar^2}{2\mu}\cdot\frac{m^2}{\rho^2} + V(\rho) = E \quad \left(\because \frac{m^2}{\rho^2}, V(\rho) \in \text{c number}\right)$$

$$\therefore F_{radial} = -\frac{\partial E}{\partial \rho} = -\frac{\partial}{\partial \rho}\left(\frac{\hbar^2}{2\mu}\frac{m^2}{\rho^2} + V(\rho)\right)$$

$$= \underbrace{\frac{\hbar^2}{\mu}\cdot\frac{m^2}{\rho^3}}_{\text{遠心力}} \underbrace{- \frac{\partial V(\rho)}{\partial \rho}}_{\text{半径方向ポテンシャル力}}$$

//

Exercise 12.4.2

(1)
$$R(\varepsilon_x \boldsymbol{i}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & -\sin\phi_x \\ 0 & \sin\phi_x & \cos\phi_x \end{bmatrix}, \quad R(\varepsilon_y \boldsymbol{j}) = \begin{bmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{bmatrix}$$

$$R(-\varepsilon_y \boldsymbol{j})\, R(-\varepsilon_x \boldsymbol{i})\, R(\varepsilon_y \boldsymbol{j})\, R(\varepsilon_x \boldsymbol{i})$$

$$= \begin{bmatrix} \cos\phi_y & 0 & \sin\phi_y \\ 0 & 1 & 0 \\ -\sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & \sin\phi_x \\ 0 & -\sin\phi_x & \cos\phi_x \end{bmatrix} \begin{bmatrix} \cos\phi_y & 0 & -\sin\phi_y \\ 0 & 1 & 0 \\ \sin\phi_y & 0 & \cos\phi_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_x & -\sin\phi_x \\ 0 & \sin\phi_x & \cos\phi_x \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1 & 0 & \varepsilon_y \\ 0 & 1 & 0 \\ -\varepsilon_y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon_x \\ 0 & -\varepsilon_x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\varepsilon_y \\ 0 & 1 & 0 \\ \varepsilon_y & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon_x \\ 0 & \varepsilon_x & 1 \end{bmatrix} = ---$$

$$= \begin{bmatrix} 1+\varepsilon_y^2 & -\varepsilon_x\varepsilon_y & -\varepsilon_y+\varepsilon_y(1+\varepsilon_x^2) \\ \varepsilon_x\varepsilon_y & 1+\varepsilon_x^2 & 0 \\ 0 & -\varepsilon_x\varepsilon_y^2 & \varepsilon_y^2+1+\varepsilon_x^2 \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1 & -\varepsilon_x\varepsilon_y & 0 \\ \varepsilon_x\varepsilon_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \simeq \begin{bmatrix} \cos\phi_x\phi_y & -\sin\phi_x\phi_y & 0 \\ \sin\phi_x\phi_y & \cos\phi_x\phi_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R(-\varepsilon_x\varepsilon_y \boldsymbol{k})$$

(2) はウラ.

(2)

$$R(-\varepsilon_y \mathbf{j})R(-\varepsilon_x \mathbf{i})R(\varepsilon_y \mathbf{j})R(\varepsilon_x \mathbf{i}) = R(-\varepsilon_x\varepsilon_y \mathbf{k})$$

$$\Longleftrightarrow \left(\mathbb{I}+\frac{i}{\hbar}\varepsilon_y L_y\right)\left(\mathbb{I}+\frac{i}{\hbar}\varepsilon_x L_x\right)\left(\mathbb{I}-\frac{i}{\hbar}\varepsilon_y L_y\right)\left(\mathbb{I}-\frac{i}{\hbar}\varepsilon_x L_x\right)$$

$$=\left(\mathbb{I}+\frac{i}{\hbar}\varepsilon_x L_x+\frac{i}{\hbar}\varepsilon_y L_y+\frac{i^2}{\hbar^2}\varepsilon_x\varepsilon_y L_y L_x\right)\left(\mathbb{I}-\frac{i}{\hbar}\varepsilon_x L_x-\frac{i}{\hbar}\varepsilon_y L_y+\frac{i^2}{\hbar^2}\varepsilon_x\varepsilon_y L_y L_x\right)$$

$$=\mathbb{I}\cancel{-\frac{i}{\hbar}\varepsilon_x L_x}\cancel{-\frac{i}{\hbar}\varepsilon_y L_y}+\frac{i^2}{\hbar^2}\varepsilon_x\varepsilon_y L_y L_x$$

$$\cancel{+\frac{i}{\hbar}\varepsilon_x L_x}\cancel{-\frac{i^2}{\hbar^2}\varepsilon_x^2 L_x^2}-\frac{i^2}{\hbar^2}\varepsilon_x\varepsilon_y L_x L_y\cancel{+\frac{i^3}{\hbar^3}\varepsilon_x^2\varepsilon_y L_x L_y L_x}$$

$$\cancel{+\frac{i}{\hbar}\varepsilon_y L_y}\cancel{-\frac{i^2}{\hbar^2}\varepsilon_x\varepsilon_y L_y L_x}\cancel{-\frac{i^2}{\hbar^2}\varepsilon_y^2 L_y^2}\cancel{+\frac{i^3}{\hbar^3}\varepsilon_x\varepsilon_y^2 L_y^2 L_x}$$

$$\cancel{+\frac{i^2}{\hbar^2}\varepsilon_x\varepsilon_y L_y L_x}\cancel{-\frac{i^3}{\hbar^3}\varepsilon_x^2\varepsilon_y L_y L_x^2}\cancel{-\frac{i^3}{\hbar^3}\varepsilon_x\varepsilon_y^2 L_y L_x L_y}$$

$$\cancel{+\frac{i^4}{\hbar^4}\varepsilon_x^2\varepsilon_y^2 L_y L_x L_y L_x}$$

$O(\varepsilon_x^2) = O(\varepsilon_y^2) = 0$として

$$\left.\begin{aligned}&= \mathbb{I}-\frac{1}{\hbar^2}\varepsilon_x\varepsilon_y(L_yL_x-L_xL_y)\\ &= \mathbb{I}+\frac{i}{\hbar}\varepsilon_x\varepsilon_y L_z\end{aligned}\right\} \cdots (*)$$

$(*)$を比較して. $(L_yL_x-L_xL_y) = -i\hbar L_z$

$$\therefore\ [L_x, L_y] = i\hbar L_z$$

Exercise 12.5.1

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow[\varepsilon_z]{\text{反時計回り}} \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix} \xrightarrow[\varepsilon_z]{\text{反時計回り}} \begin{pmatrix} \psi_x' \\ \psi_y' \end{pmatrix}$$

$$\psi_x \to \psi_x'(x, y) = \psi_x(x + y\varepsilon_z, y - x\varepsilon_z) - \psi_y(x + y\varepsilon_z, y - x\varepsilon_z)\varepsilon_z$$

$$\psi_y \to \psi_y'(x, y) = \psi_x(x + y\varepsilon_z, y - x\varepsilon_z)\varepsilon_z + \psi_y(x + y\varepsilon_z, y - x\varepsilon_z)$$

$$\langle x, y | I - \frac{i\varepsilon_z L_z}{\hbar} | \psi \rangle = \psi(x + y\varepsilon_z, y - x\varepsilon_z) \qquad (12.2.9)$$

$$\left(U[R(\varepsilon_z \mathbf{k})] \equiv I - \frac{i\varepsilon_z L_z}{\hbar}, \quad \varepsilon_z \simeq \sin\phi_0 \right)$$

を用いると.

$$\psi_x(x + y\varepsilon_z, y - x\varepsilon_z) - \psi_y(x + y\varepsilon_z, y - x\varepsilon_z)\varepsilon_z = \langle x, y | (\psi_x - \psi_y \varepsilon_z) - \frac{i\varepsilon_z L_z}{\hbar}\psi_x | \psi \rangle$$

$$\psi_x(x + y\varepsilon_z, y - x\varepsilon_z)\varepsilon_z + \psi_y(x + y\varepsilon_z, y - x\varepsilon_z) = \langle x, y | (\psi_x \varepsilon_z + \psi_y) - \frac{i\varepsilon_z L_z}{\hbar}\psi_y | \psi \rangle$$

(*)

$$\Longleftrightarrow \begin{bmatrix} \psi_x' \\ \psi_y' \end{bmatrix} = \begin{bmatrix} \psi_x - \frac{i\varepsilon_z L_z}{\hbar}\psi_x - \psi_y\varepsilon_z \\ \psi_y - \frac{i\varepsilon_z L_z}{\hbar}\psi_y + \psi_x\varepsilon_z \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \frac{i\varepsilon_z}{\hbar} \overset{\text{オービタル}}{\begin{bmatrix} L_z & 0 \\ 0 & L_z \end{bmatrix}} - \frac{i\varepsilon_z}{\hbar} \overset{\text{スピン}}{\begin{bmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{bmatrix}} \right) \begin{bmatrix} \psi_x \\ \psi_y \end{bmatrix}$$

Note

Operation (*) is equivalent to:

$$J_z = L_z^{(1)} \otimes I^{(2)} + I^{(1)} \otimes S_z^{(2)}$$

$$\equiv L_z + S_z$$

$$(\because [L, S] = 0)$$

Exercise 12.5.2

(1) $J_x^{(1/2)} = \begin{bmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{bmatrix}$, $J_y^{(1/2)} = \begin{bmatrix} 0 & i\hbar/2 \\ i\hbar/2 & 0 \end{bmatrix}$

$$[J_x^{(1/2)}, J_y^{(1/2)}] = \frac{\hbar^2}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \frac{\hbar^2}{4}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i\hbar^2/2 & 0 \\ 0 & -i\hbar^2/2 \end{bmatrix} = i\hbar\begin{bmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{bmatrix} = i\hbar J_z^{(1/2)}$$

(2) $J_x^{(1)} = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $J_y^{(1)} = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$

$$[J_x^{(1/2)}, J_y^{(1/2)}] = \frac{\hbar^2}{2}\left\{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right\}$$

$$= \frac{\hbar^2}{2}\left\{\begin{bmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & i \end{bmatrix}\right\}$$

$$= \frac{\hbar^2}{2}\begin{pmatrix} 2i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2i \end{pmatrix} = i\hbar\begin{bmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{bmatrix}$$

$= i\hbar J_z^{(1)}$ つまりブロック毎に交換関係を満たしている。

$$J_z \rightarrow \left[\begin{array}{c|cc|ccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & & & \\ \hline 0 & \hbar/2 & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & -\hbar/2 & 0 & 0 & 0 & & & & \\ \hline 0 & 0 & 0 & \hbar & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & 0 & -\hbar & & & & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 3\hbar/2 & 0 & 0 & 0 \\ \vdots & & & & & & 0 & \hbar/2 & 0 & 0 \\ \vdots & & & & & & 0 & 0 & -\hbar/2 & 0 \\ \vdots & & & & & & 0 & 0 & 0 & -3\hbar/2 \\ \hline & & & & & & & & & & \ddots \end{array}\right]$$

Exercise 12.5.3

(1) $J_x|jm\rangle = \frac{J_+ + J_-}{2}|jm\rangle = \frac{1}{2}\{c_+|j,m+1\rangle + c_-|j,m-1\rangle\}$

なので、J_xは固有値を持たない。

$$\therefore \langle J_x\rangle = \langle\psi|J_x|\psi\rangle = \sum_{j,m}\langle\psi|\underbrace{J_x|jm\rangle}_{=\frac{1}{2}\{c_+|j,m+1\rangle + c_-|j,m-1\rangle\}}\langle jm|\psi\rangle = 0$$

$\langle J_y\rangle$も同様に0.

(2) (12.5.23)と(12.5.24)から、J_x^2とJ_y^2の対角成分は等しい。

すなわち、$\langle J_x^2\rangle = \langle J_y^2\rangle$

(12.5.17a)と(12.5.17b)を用いて、

$$\langle J_x^2\rangle = \langle J_y^2\rangle = \frac{1}{2}\langle J^2 - J_z^2\rangle$$

$$= \frac{1}{2}\sum_{j,m}\langle\psi|J^2 - J_z^2|jm\rangle\langle jm|\psi\rangle$$

$$= \frac{1}{2}\sum_{j,m}\langle\psi|jm\rangle\langle jm|\psi\rangle\{j(j+1)\hbar^2 - m^2\hbar^2\}$$

$$= \frac{1}{2}\sum_{j,m}|\langle\psi|jm\rangle|^2\cdot\{j(j+1)\hbar^2 - m^2\hbar^2\}$$

$$= \frac{1}{2}\hbar^2\{j(j+1) - m^2\}$$

Exercise 12.6.2

$$R_{E\ell} = \frac{U_{E\ell}}{r}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\left\{\frac{U_{E\ell}}{r}\right\} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \left\{-\frac{U_{E\ell}}{r^2} + \frac{1}{r}\frac{dU_{E\ell}}{dr}\right\}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left\{-U_{E\ell} + r\frac{dU_{E\ell}}{dr}\right\} = \frac{1}{r^2}\left\{\cancel{-\frac{dU_{E\ell}}{dr}} + \cancel{\frac{dU_{E\ell}}{dr}} + r\frac{d^2U_{E\ell}}{dr^2}\right\}$$

$$= \frac{1}{r}\frac{d^2U_{E\ell}}{dr^2}$$

(12.6.3)に代入して

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r}\frac{d^2U_{E\ell}}{dr^2} - \frac{\ell(\ell+1)}{r^2}\cdot R_{E\ell}\right] + V(r)R_{E\ell} = E R_{E\ell}$$

両辺にrをかけて

$$\left\{-\frac{\hbar^2}{2\mu}\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2}\right] + V(r)\right\} = E$$

$$\Longleftrightarrow \quad \frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2}\{E - V(r)\} = 0$$

$$\Longleftrightarrow \quad \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right] = 0$$

Eq.(13.1.11)の導出 (Exercise 13.1.1)

$$v = \rho^{\ell+1}\sum_{k=0}^{\infty} c_k \rho^k, \quad \frac{dv}{d\rho} = (\ell+1)\rho^{\ell}\sum_{k=0}^{\infty} c_k\rho^k + \rho^{\ell+1}\sum_{k=0}^{\infty} k c_k \rho^{k-1}$$

$$\frac{d^2 v}{d\rho^2} = \ell(\ell+1)\rho^{\ell-1}\sum_{k=0}^{\infty} c_k\rho^k + (\ell+1)\rho^{\ell}\sum_{k=0}^{\infty} k c_k\rho^{k-1}$$
$$+ (\ell+1)\rho^{\ell}\sum_{k=0}^{\infty} k c_k \rho^{k-1} + \rho^{\ell+1}\sum_{k=0}^{\infty} k(k-1) c_k \rho^{k-2}$$

これらを(13.1.8)に代入して

$$\ell(\ell+1)\rho^{\ell-1}\sum_{k=0}^{\infty} c_k\rho^k + 2(\ell+1)\rho^{\ell}\sum_{k=0}^{\infty} k c_k\rho^{k-1} + \rho^{\ell+1}\sum_{k=0}^{\infty} k(k-1) c_k\rho^{k-2}$$
$$-2(\ell+1)\rho^{\ell}\sum_{k=0}^{\infty} c_k\rho^k - 2\rho^{\ell+1}\sum_{k=0}^{\infty} k c_k\rho^{k-1} + \left[\frac{e^2\lambda}{\rho} - \frac{\ell(\ell+1)}{\rho^2}\right]\rho^{\ell+1}\sum_{k=0}^{\infty} c_k\rho^k = 0$$

$$\Longleftrightarrow \left\{\cancel{\frac{\ell(\ell+1)}{\rho}} - 2(\ell+1) + e^2\lambda - \cancel{\frac{\ell(\ell+1)}{\rho}}\right\}\sum_{k=0}^{\infty} c_k\rho^k$$
$$+ \left\{2(\ell+1) - 2\rho\right\}\sum_{k=0}^{\infty} k c_k\rho^{k-1} + \rho\sum_{k=0}^{\infty} k(k-1) c_k\rho^{k-2} = 0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty}\left[\left\{e^2\lambda - 2(\ell+1)\right\}c_k\rho^k + 2(\ell+1)k c_k\rho^{k-1} - 2k c_k\rho^k + k(k-1)c_k\rho^{k-1}\right] = 0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty}\left[\left\{e^2\lambda - 2(k+\ell+1)\right\}c_k\rho^k + (2\ell+2+k-1)k c_k\rho^{k-1}\right] = 0$$

$$\Longleftrightarrow \sum_{k=0}^{\infty}\left[\left\{e^2\lambda - 2(k+\ell+1)\right\}c_k\rho^k + \underbrace{(2\ell+2+k)(k+1)}_{(k+\ell+2)(k+\ell+1)-\ell(\ell+1)} c_{k+1}\rho^k\right] = 0$$

$$\therefore \quad \frac{c_{k+1}}{c_k} = \frac{-e^2\lambda + 2(k+\ell+1)}{(k+\ell+2)(k+\ell+1) - \ell(\ell+1)}$$

Exercise 13.1.2

P.326, (12.5.8)より, ある n に対し

$$l = 0, 1, \cdots, n, \quad (\leftarrow n個)$$

$$m = -l, \cdots, 0, \cdots, +l \quad (\leftarrow 2l+1個)$$

(13.1.15)より, $n = k + l + 1$, すなわち

$$\left.\begin{array}{cc} k & l \\ \hline 0 & n-1 \\ 1 & n-2 \\ \vdots & \vdots \\ n-1 & 0 \end{array}\right\} n個$$

各 l について, $2l+1$ 個の m が存在するので.

$$(\#\ \text{of degeneracy}) = \sum_{l=0}^{n-1} (2l+1) = n^2$$

Exercise 13.1.5　(量子力学における、ビリアルの定理)

$\Omega = PR$ として、エーレンフェストの定理

$\frac{d}{dt}\langle\Omega\rangle = -\frac{i}{\hbar}\langle[\Omega, H]\rangle$ および $\frac{d}{dt}\langle PR\rangle = 0$ を用いる

また極座標での半径方向運動量演算子は、

$$P_R = -i\hbar\left(\frac{\partial}{\partial R} + \frac{1}{2R}\right) \quad \cdots(\text{エルミート演算、P.216参照})$$

である。

$$\frac{d}{dt}\langle PR\rangle = -\frac{i}{\hbar}\langle[PR, H]\rangle = 0 \iff [H, PR] = 0$$

$$\iff \langle P[H,R] + [H,P]R\rangle = 0 \quad \cdots ① \qquad \because [\Omega, \Lambda\theta] = \Lambda[\Omega, \theta] + [\Omega, \Lambda]\theta \ (1.5.10)$$

$$\langle H\rangle = \langle T\rangle + \langle V\rangle = \left\langle\frac{P^2}{2m}\right\rangle - \left\langle\frac{e^2}{r}\right\rangle$$

$r \to R$, $P \to P_R$ として

$$[H, R] = \left(\frac{P^2}{2m} - \frac{e^2}{R}\right)R - R\left(\frac{P^2}{2m} - \frac{e^2}{R}\right) = \frac{1}{2m}[P^2, R] \quad \cdots ②$$

$$[P^2, R] = [PP, R] = -[R, PP] = -P[R,P] - [R,P]P \quad \cdots ③$$

ここで、

$$[P,R]\psi = -i\hbar\left(\frac{\partial}{\partial R} + \frac{1}{2R}\right)R\psi + ri\hbar\left(\frac{\partial}{\partial R} + \frac{1}{2R}\right)\psi$$

$$= -i\hbar\left\{\psi + R\frac{\partial\psi}{\partial R} + \frac{1}{2}\psi\right\} + Ri\hbar\left\{\frac{\partial\psi}{\partial R} + \frac{1}{2R}\psi\right\}$$

$$= i\hbar\left\{-\frac{3}{2} - R\frac{\partial}{\partial R} + R\frac{\partial}{\partial R} + \frac{1}{2}\right\}\psi = -i\hbar\psi$$

$$\therefore [P, R] = -i\hbar \quad \cdots ④$$

④を③に代入し、$[P^2, R] = -2i\hbar P$

$\therefore$ ②より $[H, R] = \frac{i\hbar P}{m}$ 　$\therefore P[H,R] = -i\hbar\frac{P^2}{m} \quad \cdots ⑤$

(ウラ面へ)

$$[H, P] = \left(\frac{P^2}{2m} - \frac{e^2}{R}\right)P - P\left(\frac{P^2}{2m} - \frac{e^2}{R}\right) = -\frac{e^2}{R}P + P\frac{e^2}{R}$$

$$= e^2\left[P, \frac{1}{R}\right] \quad \cdots ⑥$$

$$\left[P, \frac{1}{R}\right]\psi = -i\hbar\left(\frac{\partial}{\partial R} + \frac{1}{2R}\right)\frac{1}{R}\psi + i\hbar\frac{1}{R}\left(\frac{\partial}{\partial R} + \frac{1}{2R}\right)\psi$$

$$= -i\hbar\left(-\frac{1}{R^2}\psi + \cancel{\frac{1}{R}\frac{\partial\psi}{\partial R}} + \cancel{\frac{1}{2R^2}\psi}\right) + i\hbar\left(\cancel{\frac{1}{R}\frac{\partial\psi}{\partial R}} + \cancel{\frac{1}{2R^2}\psi}\right)$$

$$= i\hbar\cdot\frac{1}{R^2}\psi$$

$\therefore \left[P, \frac{1}{R}\right] = i\hbar\cdot\frac{1}{R^2}$　これを⑥に代入し、$\therefore [H, P] = i\hbar\frac{e^2}{R^2}$

$$\therefore [H, P]R = i\hbar\frac{e^2}{R} \quad \cdots ⑦$$

⑤、⑦を①に代入して $R \to r$ として

$$\left\langle -i\hbar\frac{P^2}{m} + i\hbar\frac{e^2}{r}\right\rangle = 0$$

$$\Longleftrightarrow \left\langle 2\cdot\frac{P^2}{2m}\right\rangle = \left\langle\frac{e^2}{r}\right\rangle$$

$\langle T\rangle = \left\langle\frac{P^2}{2m}\right\rangle$, $\langle V\rangle = -\left\langle\frac{e^2}{r}\right\rangle$ より

$$\langle T\rangle = -\frac{1}{2}\langle V\rangle$$

これは、$V(r) = cr^K$　$(K = -1)$ のときの

ビリアルの定理 $\bar{T} = \frac{K}{2}\bar{V}$ の量子力学表現になっている。

Given this, how could one forget that the levels go as n^{-2}, i.e.,

$$E_n = -\frac{E_1}{n^2}?$$

If we rewrite E_1 as $-e^2/2a_0$, we can get the formula for a_0. The equation $\alpha = \beta$ also justifies the use of nonrelativistic quantum mechanics. An equivalent way (which avoids the use of velocity) is Eq. (13.3.17), which states that the binding energy is $\simeq (1/137)^2$ times the rest energy of the electron.

√ *Exercise 13.3.1.** The pion has a range of 1 Fermi $= 10^{-5}$ Å as a mediator of nuclear force. Estimate its rest energy.

$E \approx \frac{\hbar c}{\Delta X} = \frac{2000\ \mathrm{eV \cdot Å}}{10^{-5}\ \mathrm{Å}} = 200\ \mathrm{MeV}$

√ *Exercise 13.3.2.** Estimate the de Broglie wavelength of an electron of kinetic energy 200 eV. (Recall $\lambda = 2\pi\hbar/p$.)

$E = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m} \Leftrightarrow p = \sqrt{2mE}$

$\therefore \lambda = \frac{2\pi\hbar}{\sqrt{2mE}} = \frac{2\pi\hbar c}{\sqrt{2mc^2E}} \simeq \frac{6 \times 2000\ \mathrm{eV \cdot Å}}{\sqrt{1\ \mathrm{MeV} \cdot 200\ \mathrm{eV}}} = \frac{1.2 \times 10^4}{1.4 \times 10^4} \simeq 1\ \mathrm{Å}$

Comparison with Experiment

Quantum theory makes very detailed predictions for the hydrogen atom. Let us ask how these are to be compared with experiment. Let us consider first the energy levels and then the wave functions. In principle, one can measure the energy levels by simply weighing the atom. In practice, one measures the *differences in energy levels* as follows. If we start with the atom in an eigenstate $|nlm\rangle$, it will stay that way forever. However, if we perturb it for a time T, by turning on some external field (i.e., change the Hamiltonian from H^0, the Coulomb Hamiltonian, to $H^0 + H^1$) its state vector can start moving around in Hilbert space, since $|nlm\rangle$ is not a stationary state of $H^0 + H^1$. If we measure the energy at time $t > T$, we may find it corresponds to another state with $n' \neq n$. One measures the energy by detecting the photon emitted by the atom. The frequency of the detected photon will be

$$\omega_{nn'} = \frac{E_n - E_{n'}}{\hbar} \tag{13.3.18}$$

Thus the frequency of light coming out of hydrogen will be

Remember,
$E_n = -\frac{Ry}{n^2}$ (13.1.20)

$$\begin{aligned}\omega_{nn'} &= \frac{\mathrm{Ry}}{\hbar}\left(-\frac{1}{n^2} + \frac{1}{n'^2}\right) \\ &= \frac{\mathrm{Ry}}{\hbar}\left(\frac{1}{n'^2} - \frac{1}{n^2}\right)\end{aligned} \tag{13.3.19}$$

基底状態への遷移

For a fixed value $n' = 1, 2, 3, \ldots$, we obtain a family of lines as we vary n. These families have in fact been seen, at least for several values of n'. The $n' = 1$ family is

Exercise 14.3.2 (for proof of eqs. 14.3.28 & 14.3.29)

(1)
$$\hat{n}\cdot S = \frac{\hbar}{2}\begin{bmatrix}\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta\end{bmatrix} \equiv A$$

$$A x = a x \iff (A - a\mathbb{I})x = 0$$

有意な固有値が存在する為には、$\det(A - a\mathbb{I}) = 0$

$$\iff |A - a\mathbb{I}| = -\left(\frac{\hbar}{2}\cos\theta - a\right)\left(\frac{\hbar}{2}\cos\theta + a\right) - \frac{\hbar}{2}\sin\theta e^{-i\phi}\frac{\hbar}{2}\sin\theta e^{i\phi}$$

$$= -\left(\frac{\hbar^2}{4}\cos^2\theta - a^2\right) - \frac{\hbar^2}{4}\sin^2\theta = a^2 - \frac{\hbar^2}{4} = 0$$

固有値は $a = \pm\frac{\hbar}{2}$.

(i) $a = \frac{\hbar}{2}$ について.

$$\left(A - \frac{\hbar}{2}\mathbb{I}\right)x = \frac{\hbar}{2}\begin{bmatrix}\cos\theta - 1 & \sin\theta\cdot e^{-i\phi} \\ \sin\theta\cdot e^{i\phi} & -\cos\theta - 1\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}$$

$$= \frac{\hbar}{2}\begin{bmatrix}(\cos\theta - 1)x + \sin\theta\cdot e^{-i\phi} y \\ \sin\theta\cdot e^{i\phi}\cdot x - (\cos\theta + 1) y\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

この連立方程式は、

第1式 $\Rightarrow \dfrac{y}{x} = \dfrac{\sin\theta e^{i\phi}}{1+\cos\theta}$, 第2式 $\Rightarrow \dfrac{y}{x} = \dfrac{1-\cos\theta e^{i\phi}}{\sin\theta}$

で、$\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$ なので、第1式と第2式は同一.

$$\frac{y}{x} = \frac{\sin\theta e^{i\phi}}{1+\cos\theta} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}}\, e^{i\phi} = \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}\, e^{i\phi}$$

$$= \frac{\sin\frac{\theta}{2}\cdot e^{i\phi/2}}{\cos\frac{\theta}{2}\cdot e^{-i\phi/2}} \qquad \therefore |\hat{n}\uparrow\rangle \equiv |\hat{n}+\rangle = \begin{bmatrix}\cos\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ \sin\left(\frac{\theta}{2}\right) e^{i\phi/2}\end{bmatrix}$$

(ii) $a = -\frac{\hbar}{2}$ について

(i) のときと同様にして、$|\hat{n}\downarrow\rangle \equiv |\hat{n}-\rangle = \begin{bmatrix}-\sin\left(\frac{\theta}{2}\right) e^{-i\phi/2} \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi/2}\end{bmatrix}$

Exercise 14.3.2 (cont.)

(2)

$$\mathbb{S} = S_x \mathbb{i} + S_y \mathbb{j} + S_z \mathbb{k} = \frac{\hbar}{2}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbb{i} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\mathbb{j} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbb{k}\right\}$$

$$= \begin{bmatrix} \mathbb{k} & \mathbb{i} - i\mathbb{j} \\ \mathbb{i} + i\mathbb{j} & -\mathbb{k} \end{bmatrix}$$

まず、$|\hat{n}+\rangle = \begin{bmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{bmatrix}$ (14.3.28a) の場合については、

$$\langle \hat{n}+|\mathbb{S}|\hat{n}+\rangle = \frac{\hbar}{2}\left[\cos\left(\frac{\theta}{2}\right)e^{i\phi/2},\ \sin\left(\frac{\theta}{2}\right)e^{-i\phi/2}\right]\begin{bmatrix} \mathbb{k} & \mathbb{i} - i\mathbb{j} \\ \mathbb{i} + i\mathbb{j} & -\mathbb{k} \end{bmatrix}\begin{bmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2} \\ \sin\left(\frac{\theta}{2}\right)e^{i\phi/2} \end{bmatrix}$$

$$= \frac{\hbar}{2}\left[\cos\left(\frac{\theta}{2}\right)e^{i\phi/2},\ \sin\left(\frac{\theta}{2}\right)e^{-i\phi/2}\right]\begin{bmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}\cdot\mathbb{k} + \sin\left(\frac{\theta}{2}\right)e^{i\phi/2}(\mathbb{i} - i\mathbb{j}) \\ \cos\left(\frac{\theta}{2}\right)e^{-i\phi/2}\cdot(\mathbb{i} + i\mathbb{j}) - \sin\left(\frac{\theta}{2}\right)e^{i\phi/2}\cdot\mathbb{k} \end{bmatrix}$$

$$= \frac{\hbar}{2}\left[\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{i\phi}(\mathbb{i} - i\mathbb{j}) + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i\phi}(\mathbb{i} + i\mathbb{j}) + \cos^2\left(\frac{\theta}{2}\right)\cdot\mathbb{k} - \sin^2\left(\frac{\theta}{2}\right)\cdot\mathbb{k}\right]$$

$$= \frac{\hbar}{2}\left[2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\cdot\frac{e^{i\phi}+e^{-i\phi}}{2}\cdot\mathbb{i} - 2i\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\frac{e^{i\phi}-e^{-i\phi}}{2}\cdot\mathbb{j} + \left\{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\right\}\cdot\mathbb{k}\right]$$

$$= \frac{\hbar}{2}\left[\sin\theta\cos\phi\cdot\mathbb{i} + \sin\theta\cdot\sin\phi\cdot\mathbb{j} + \cos\theta\cdot\mathbb{k}\right]$$

$$= \frac{\hbar}{2}\left\{n_x\cdot\mathbb{i} + n_y\cdot\mathbb{j} + n_z\cdot\mathbb{k}\right\} = \frac{\hbar}{2}\hat{n}$$

$|\hat{n}-\rangle$の場合も同様.

Exercise 14.3.3

$$\mathrm{Tr}\,\sigma_i = \sum_l \langle l|\sigma_i|l\rangle = \sum_l \langle l|-i\sigma_j\sigma_k|l\rangle$$

$$\because\ \sigma_j\sigma_k = i\sigma_i \quad (14.3.33)$$

$$\sigma_j \equiv \begin{bmatrix} \sigma_{ja} & \sigma_{jb} \\ \sigma_{jc} & \sigma_{jd} \end{bmatrix}, \quad \sigma_k \equiv \begin{bmatrix} \sigma_{ka} & \sigma_{kb} \\ \sigma_{kc} & \sigma_{kd} \end{bmatrix}$$ とおき、

$$[\sigma_j, \sigma_k]_+ = \sigma_j\sigma_k + \sigma_k\sigma_j = 0 \quad (14.3.32)$$ に適用すると

$$\sigma_j\sigma_k = \begin{bmatrix} \sigma_{ja} & \sigma_{jb} \\ \sigma_{jc} & \sigma_{jd} \end{bmatrix}\begin{bmatrix} \sigma_{ka} & \sigma_{kb} \\ \sigma_{kc} & \sigma_{kd} \end{bmatrix} = \begin{bmatrix} \sigma_{ja}\sigma_{ka}+\sigma_{jb}\sigma_{kc} & \sigma_{ja}\sigma_{kb}+\sigma_{jb}\sigma_{kd} \\ \sigma_{jc}\sigma_{ka}+\sigma_{jd}\sigma_{kc} & \sigma_{jc}\sigma_{kb}+\sigma_{jd}\sigma_{kd} \end{bmatrix}$$

$$-\sigma_k\sigma_j = -\begin{bmatrix} \sigma_{ka} & \sigma_{kb} \\ \sigma_{kc} & \sigma_{kd} \end{bmatrix}\begin{bmatrix} \sigma_{ja} & \sigma_{jb} \\ \sigma_{jc} & \sigma_{jd} \end{bmatrix} = -\begin{bmatrix} \sigma_{ka}\sigma_{ja}+\sigma_{kb}\sigma_{jc} & \sigma_{ka}\sigma_{jb}+\sigma_{kb}\sigma_{jd} \\ \sigma_{kc}\sigma_{ja}+\sigma_{kd}\sigma_{jc} & \sigma_{kc}\sigma_{jb}+\sigma_{kd}\sigma_{jd} \end{bmatrix}$$

(1,1) 要素を比較して、

$$\left.\begin{array}{l}
\sigma_{ja}\sigma_{ka}+\sigma_{jb}\sigma_{kc} = -\sigma_{ka}\sigma_{ja}-\sigma_{kb}\sigma_{jc} \\
\iff \quad 2\sigma_{ja}\sigma_{ka}+\sigma_{jb}\sigma_{kc}+\sigma_{kb}\sigma_{jc} = 0 \\
\text{(2,2) 要素を比較して} \\
\sigma_{jc}\sigma_{kb}+\sigma_{jd}\sigma_{kd} = -\sigma_{kc}\sigma_{jb}-\sigma_{kd}\sigma_{jd} \\
\iff \quad 2\sigma_{jd}\sigma_{kd}+\sigma_{jc}\sigma_{kb}+\sigma_{kc}\sigma_{jb} = 0
\end{array}\right\} \cdots (*)$$

$$\begin{aligned}
\therefore\ \mathrm{Tr}\,\sigma_i &= \sum_l \langle l|-i\sigma_j\sigma_k|l\rangle \\
&= -i(\sigma_{ja}\sigma_{ka}+\sigma_{jb}\sigma_{kc}+\sigma_{jc}\sigma_{kb}+\sigma_{jd}\sigma_{kd}) \\
&= -i\Big\{-\tfrac{1}{2}(\sigma_{jb}\sigma_{kc}+\sigma_{kb}\sigma_{jc})+\sigma_{jb}\sigma_{kc} \\
&\qquad +\sigma_{jc}\sigma_{kb}-\tfrac{1}{2}(\sigma_{jc}\sigma_{kb}+\sigma_{kc}\sigma_{jb})\Big\} \quad \because (*) \\
&= 0
\end{aligned}$$

Exercise 15.1.1

$$S^2 = S_1{}^2 + S_2{}^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

$$(S_1{}^2 + S_2{}^2)|++\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|++\rangle = \frac{3}{2}\hbar^2|++\rangle$$

$$2S_{1z}S_{2z}|++\rangle = 2\cdot\frac{\hbar}{2}\cdot\frac{\hbar}{2}|++\rangle = \frac{\hbar^2}{2}|++\rangle$$

$S_{1+}S_{2-}|++\rangle = 0$, $S_{1-}S_{2+}|++\rangle = 0$ $\because$ $S_{1+}|++\rangle = 0$ (See, P.323)

$$\therefore S^2|++\rangle = \left(\frac{3}{2}\hbar^2 + \frac{1}{2}\hbar^2\right)|++\rangle = 2\hbar^2|++\rangle$$

同様にして、 $S^2|--\rangle = \left(\frac{3}{2}\hbar^2 + \frac{1}{2}\hbar^2\right)|--\rangle = 2\hbar^2|--\rangle$

$$(S_1{}^2 + S_2{}^2)|+-\rangle = \left(\frac{3}{4}\hbar^2 + \frac{3}{4}\hbar^2\right)|+-\rangle = \frac{3}{2}\hbar^2|+-\rangle$$

$$2S_{1z}S_{2z}|+-\rangle = 2\cdot\frac{\hbar}{2}\left(-\frac{\hbar}{2}\right)|+-\rangle = -\frac{\hbar^2}{2}|+-\rangle$$

$S_{1+}S_{2-}|+-\rangle = 0$ $\because$ $S_{2-}|+-\rangle = 0$

$S_{1-}S_{2+}|+-\rangle$ については、(12.5.19) より

$$J_\pm|j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle$$

$$\Rightarrow S_{1-}S_{2+}|+-\rangle = \hbar\sqrt{\left(\frac{1}{2}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{1}{2}+1\right)}\cdot\hbar\sqrt{\left\{\frac{1}{2}-\left(-\frac{1}{2}\right)\right\}\left\{\frac{1}{2}+\left(-\frac{1}{2}\right)+1\right\}}\,|-+\rangle$$

$$= \hbar^2|-+\rangle$$

$$\therefore S^2|+-\rangle = \left(\frac{3}{2}\hbar^2 - \frac{1}{2}\hbar^2\right)|+-\rangle + \hbar^2|-+\rangle$$

$$= \hbar^2|+-\rangle + \hbar^2|-+\rangle$$

同様にして、 $S^2|-+\rangle = \hbar^2|-+\rangle + \hbar^2|+-\rangle$

(ウラへ)

(Exercise 15.1.1 つづき)

$$\left.\begin{aligned} S^2|++\rangle &= 2\hbar^2|++\rangle \\ S^2|+-\rangle &= \hbar^2(|+-\rangle+|-+\rangle) \\ S^2|-+\rangle &= \hbar^2(|+-\rangle+|-+\rangle) \\ S^2|--\rangle &= 2\hbar^2|--\rangle \end{aligned}\right\} \Rightarrow S^2 \xrightarrow[\text{basis}]{\text{Product}} \hbar^2\begin{bmatrix} 2&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&2\end{bmatrix}\begin{bmatrix}|++\rangle\\|+-\rangle\\|-+\rangle\\|--\rangle\end{bmatrix}$$

∴

$$S^2\frac{|+-\rangle+|-+\rangle}{\sqrt2} = \frac{1}{\sqrt2}\{S^2|+-\rangle+S^2|-+\rangle\}$$
$$= \frac{1}{\sqrt2}\{\hbar^2(|+-\rangle+|-+\rangle)+\hbar^2(|-+\rangle+|+-\rangle)\}$$
$$= 2\hbar^2\frac{|+-\rangle+|-+\rangle}{\sqrt2}$$ ∴ 固有値 $2\hbar^2$

$$S^2\frac{|+-\rangle-|-+\rangle}{\sqrt2} = \frac{1}{\sqrt2}\{S^2|+-\rangle-S^2|-+\rangle\} = 0$$ ∴ 固有値 0

$$\therefore S^2 \xrightarrow[\text{basis}]{\text{Total-}s} \hbar^2 \begin{array}{cccc} |++\rangle & \frac{|+-\rangle+|-+\rangle}{\sqrt2} & \frac{|+-\rangle-|-+\rangle}{\sqrt2} & |--\rangle \end{array}$$
$$\begin{bmatrix} 2&0&0&0\\0&2&0&0\\0&0&0&0\\0&0&0&2\end{bmatrix}$$

← 既約表現行列
(固有値で対角化されている)

$$\left.\begin{aligned} &|++\rangle \\ &\frac{|+-\rangle+|-+\rangle}{\sqrt2} \\ &|--\rangle \end{aligned}\right\}\cdots \text{固有値 } 2\hbar^2 \Leftrightarrow s(s+1)=2 \Leftrightarrow s=1$$

等価スピン

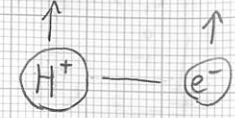
∴ Triplet (↑↑)

$$\frac{|+-\rangle-|-+\rangle}{\sqrt2}\cdots \text{固有値 } 0 \Leftrightarrow s(s+1)=0 \Leftrightarrow s=0$$

∴ Singlet (↑↓)

Exercise 15.1.2

(1)



$$\mathcal{H}_{hf} = A\,\mathbf{S}_1\cdot\mathbf{S}_2 = \frac{1}{2}A\left(S^2 - S_1{}^2 - S_2{}^2\right)$$

Exercise 15.1.1より $\quad S^2 \xrightarrow[\text{Total-}S]{} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

$$\begin{cases} S_1{}^2\,|++\rangle = \hbar^2\frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}\hbar^2 \\[2ex] S_1{}^2\,\dfrac{|+-\rangle + |-+\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\left\{\frac{3}{4}\hbar^2|+-\rangle + \frac{3}{4}\hbar^2|-+\rangle\right\} = \frac{3}{4}\hbar^2\,\dfrac{|+-\rangle + |-+\rangle}{\sqrt{2}} \\[2ex] S_1{}^2\,\dfrac{|+-\rangle - |-+\rangle}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}\left\{\frac{3}{4}\hbar^2|+-\rangle - \frac{3}{4}\hbar^2|-+\rangle\right\} = \frac{3}{4}\hbar^2\,\dfrac{|+-\rangle - |-+\rangle}{\sqrt{2}} \\[2ex] S_1{}^2\,|--\rangle = \frac{3}{4}\hbar^2|--\rangle \end{cases}$$

これらはS_2についても同様

$$\therefore\ \mathcal{H}_{hf} \xrightarrow[\substack{\text{Total-}S\\ \text{basis}}]{} \frac{1}{2}A\left\{\hbar^2\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - 2\cdot\hbar^2\begin{bmatrix} 3/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 0 \\ 0 & 0 & 3/4 & 0 \\ 0 & 0 & 0 & 3/4 \end{bmatrix}\right\}$$

$$= \frac{1}{2}A\hbar^2\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -3/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix} = A\hbar^2\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & -3/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

$|100\rangle \longrightarrow E_1 = -R_y$ (13.1.20) より.

$$\therefore\ E_+ = -R_y + \frac{A\hbar^2}{4} \quad (\text{Triplet})\ \uparrow\uparrow$$

$$E_- = -R_y - \frac{3A\hbar^2}{4} \quad (\text{Singlet})\ \uparrow\downarrow$$

この場合, Tripletの方がエネルギーが高くなっている.

Exercise 15.1.2 つづき

(2)

$$\Delta E = E_+ - E_- = \hbar^2 A$$

$\mathcal{H}_{hf} \cong \dfrac{\mu_e \cdot \mu_p}{a_0^3}$ とすると.

$$\begin{cases} \mu_e = \gamma_e S_e = -\dfrac{e}{mc} S_e & \because (14.4.18) \\ \mu_p = \gamma_p S_p = \dfrac{g_p e}{2Mc} S_p & \text{ここで } g_p = 5.6 \text{ (P. 391参照)} \end{cases}$$

$\therefore \mathcal{H}_{hf} \cong -\dfrac{e}{mc}\dfrac{g_p e}{2Mc}\cdot\dfrac{1}{a_0^3}\cdot S_e\cdot S_p$ ここで $S_1 = S_e$, $S_2 = S_p$ なので,

$$A \cong \frac{e^2}{mc}\cdot\frac{5.6e}{2Mc}\cdot\frac{1}{a_0^3}$$

以下、$R_y = \dfrac{me^4}{2\hbar^2}$ (13.1.19), $a_0 = \dfrac{\hbar^2}{me^2}$ (13.1.24), $\alpha = \dfrac{e^2}{\hbar c}$ (13.3.7) を使う.

$$\Delta E = A\hbar^2 \cong \frac{e}{mc}\cdot\frac{5.6e}{2Mc}\cdot\frac{\hbar^2}{a_0^3} = \frac{2.8e^2\hbar^2}{mMc^2a_0^3} = \frac{2.8e^2\hbar^2}{mMc^2}\cdot\frac{m^3e^6}{\hbar^6}$$

$$= \frac{2.8e^8m^2}{Mc^2\hbar^4} = \frac{2\cdot 2.8e^4m}{Mc^2\hbar^2}\cdot\frac{me^4}{2\hbar^2} = \frac{5.6me^4}{Mc^2\hbar^2}\cdot R_y$$

$$= \frac{m}{M}5.6\cdot\frac{e^4}{\hbar^2c^2}\cdot R_y = 5.6\frac{m}{M}\alpha^2R_y \sim \frac{m}{M}\alpha^2R_y$$

$$\Delta E \approx 5.6\cdot\frac{m}{M}\alpha^2R_y = 5.6\cdot\frac{1}{1836}\cdot\frac{1}{(137)^2}\cdot 13.6 = 2.21\times10^{-6}\ [\mathrm{eV}]$$

$$= 2.2\ \mu\mathrm{eV} = 56.4\ \mathrm{cm}^{-1}$$

This is the same order to observed $H_+ \leftrightarrow H_-$ line of $21.4\ \mathrm{cm}^{-1}$.

(3)

$$\Delta E \ll |R_y|$$

At R.T. $k_BT = 25\ \mathrm{meV} \gg \Delta E \approx 2.2\ \mu\mathrm{eV}$

$$\therefore\ P(\text{triplet})/P(\text{singlet}) = e^{\Delta E/k_BT} \cong 1$$

Exercise 15.2.2

$$\frac{1}{2} \otimes 1 = \frac{3}{2} \oplus \frac{1}{2}$$

Product		Total	
$+\ \|1,1\rangle$	$\|\frac{1}{2},\frac{1}{2}\rangle +$	$\|\frac{3}{2},\frac{3}{2}\rangle$	
$0\ \|1,0\rangle$ $\otimes$		$\|\frac{3}{2},\frac{1}{2}\rangle$ $\oplus$	$\|\frac{1}{2},\frac{1}{2}\rangle$
$-\ \|1,-1\rangle$	$\|\frac{1}{2},-\frac{1}{2}\rangle -$	$\|\frac{3}{2},-\frac{1}{2}\rangle$	$\|\frac{1}{2},-\frac{1}{2}\rangle$
		$\|\frac{3}{2},-\frac{3}{2}\rangle$	

$$S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$$

① $S^2|++\rangle$

$$(S_1^2+S_2^2)|++\rangle = \left\{\hbar^2\cdot 1\cdot(1+1) + \hbar^2\cdot\frac{1}{2}\left(\frac{1}{2}+1\right)\right\}|++\rangle$$

$$= \left(2\hbar^2 + \frac{3}{4}\hbar^2\right)|++\rangle = \frac{11}{4}|++\rangle$$

$$2S_{1z}S_{2z}|++\rangle = 2\cdot\hbar\cdot\frac{\hbar}{2}|++\rangle = \hbar^2|++\rangle$$

$$S_{1+}S_{2-}|++\rangle = S_{1-}S_{2+}|++\rangle = 0$$

$$\therefore\ S|++\rangle = \left(\frac{11}{4}\hbar^2 + \hbar^2\right) = \frac{15}{4}\hbar^2$$

② $S^2|+-\rangle$

$$(S_1^2+S_2^2)|+-\rangle = \frac{11}{4}\hbar^2|+-\rangle$$

$$2S_{1z}S_{2z}|+-\rangle = 2\hbar\left(-\frac{\hbar}{2}\right)|+-\rangle = -\hbar^2|+-\rangle$$

$$S_{1+}S_{2-}|+-\rangle = 0.$$

$$S_{1-}S_{2+}|+-\rangle = S_{1-}|+\rangle\cdot S_{2+}|-\rangle$$

$$= \hbar\underbrace{\sqrt{(1+1)\cdot(1-1+1)}}_{=\sqrt{2}}|0\rangle\cdot\hbar\overbrace{\sqrt{\left\{\frac{1}{2}-\left(-\frac{1}{2}\right)\right\}\left(\frac{1}{2}-\frac{1}{2}+1\right)}}^{=1}|+\rangle$$

$$= \sqrt{2}\hbar^2|0+\rangle$$

$$\therefore\ S^2|+-\rangle = \left(\frac{11}{4}\hbar^2 - \hbar^2\right)|+-\rangle + \sqrt{2}\hbar^2|0+\rangle = \frac{7}{4}\hbar^2|+-\rangle + \sqrt{2}\hbar^2|0+\rangle$$

(ウラへ)

③ $\underline{S^2|0+\rangle}$

$(S_1^2+S_2^2)|0+\rangle = \frac{11}{4}\hbar^2|0+\rangle$

$2S_{1z}S_{2z}|0+\rangle = 2\hbar\cdot 0\cdot\frac{1}{2}|0+\rangle = 0$

$S_{1+}S_{2-}|0+\rangle = S_{1+}|0\rangle\cdot S_{2-}|+\rangle = \hbar\sqrt{(1-0)(1+0+1)}|+\rangle\cdot\hbar\sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-\frac{1}{2}+1)}|-\rangle$

$= \hbar^2\cdot\sqrt{2}\cdot|+-\rangle$

$S_{1-}S_{2+}|0+\rangle = 0$

$\therefore S^2|0+\rangle = \frac{11}{4}\hbar^2|0+\rangle + \sqrt{2}\hbar|+-\rangle$

④ $\underline{S^2|0-\rangle}$

$(S_1^2+S_2^2)|0-\rangle = \frac{11}{4}\hbar^2|0-\rangle$

$2S_{1z}S_{2z}|0-\rangle = 0$

$S_{1+}S_{2-}|0-\rangle = 0$

$S_{1-}S_{2+}|0-\rangle = S_{1-}|0\rangle\cdot S_{2+}|-\rangle = \hbar\sqrt{(1+0)(1-0+1)}|-\rangle\cdot\hbar|+\rangle = \sqrt{2}\hbar^2|-+\rangle$

$\therefore S^2|0-\rangle = \frac{11}{4}\hbar^2|0-\rangle + \sqrt{2}\hbar^2|-+\rangle$

⑤ $\underline{S^2|-+\rangle}$

$(S_1^2+S_2^2)|-+\rangle = \frac{11}{4}\hbar^2|-+\rangle$

$2S_{1z}S_{2z}|-+\rangle = 2(-\hbar)\cdot\frac{\hbar}{2}|-+\rangle = -\hbar^2|-+\rangle$

$S_{1+}S_{2+}|-+\rangle = S_{1+}|-\rangle\cdot S_{2-}|+\rangle = \hbar\sqrt{\{1-(-1)\}(1-1+1)}|0\rangle\cdot\hbar\sqrt{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-\frac{1}{2}+1)}|-\rangle$

$= \hbar^2\cdot\sqrt{2}|0-\rangle$

$S_{1-}S_{2+}|-+\rangle = 0$

$\therefore S^2|-+\rangle = (\frac{11}{4}\hbar^2-\hbar^2)|-+\rangle + \sqrt{2}\hbar^2|0-\rangle = \frac{7}{4}\hbar^2|-+\rangle + \sqrt{2}\hbar^2|0-\rangle$

⑥ $\underline{S^2|--\rangle}$

$(S_1^2+S_2^2)|--\rangle = \frac{11}{4}\hbar^2|--\rangle$

$2S_{1z}S_{2z}|--\rangle = 2(-\hbar)(-\frac{\hbar}{2})|--\rangle = \hbar^2|--\rangle$

$S_{1+}S_{2-}|--\rangle = S_{1-}S_{2+}|--\rangle = 0$

$\therefore S^2|--\rangle = (\frac{11}{4}\hbar^2+\hbar^2)|--\rangle = \frac{15}{4}\hbar^2|--\rangle$

（次ページへ．）

まとめると、

$S^2|++\rangle = \frac{15}{4}\hbar^2|++\rangle$

$S^2|+-\rangle = \frac{7}{4}\hbar^2|+-\rangle + \sqrt{2}\hbar^2|0+\rangle$

$S^2|0+\rangle = \frac{11}{4}\hbar^2|0+\rangle + \sqrt{2}\hbar^2|+-\rangle$

$S^2|0-\rangle = \frac{11}{4}\hbar^2|0-\rangle + \sqrt{2}\hbar^2|-+\rangle$

$S^2|-+\rangle = \frac{7}{4}\hbar^2|-+\rangle + \sqrt{2}\hbar^2|0-\rangle$

$S^2|--\rangle = \frac{15}{4}\hbar^2|--\rangle$

$\Longrightarrow$

$|\frac{3}{2},\frac{3}{2}\rangle = |++\rangle$

$|\frac{3}{2},\frac{1}{2}\rangle = \alpha_1|+-\rangle + \beta_1|0+\rangle$

$|\frac{3}{2},-\frac{1}{2}\rangle = \alpha_2|0-\rangle + \beta_2|-+\rangle$

$|\frac{3}{2},-\frac{3}{2}\rangle = |--\rangle$

$|\frac{1}{2},\frac{1}{2}\rangle = \alpha_3|+-\rangle - \beta_3|0+\rangle$

$|\frac{1}{2},-\frac{1}{2}\rangle = \alpha_4|0-\rangle - \beta_4|-+\rangle$

↑の形が推測できる。

$$\therefore\ S^2 \xrightarrow[\text{Product basis}]{} \hbar^2 \begin{bmatrix} \frac{15}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{7}{4} & \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & \frac{11}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{11}{4} & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} & \frac{7}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{4} \end{bmatrix} \begin{bmatrix} |++\rangle \\ |+-\rangle \\ |0+\rangle \\ |0-\rangle \\ |-+\rangle \\ |--\rangle \end{bmatrix} \quad \text{for } 1 \otimes \frac{1}{2}$$

上の推測から、

$$S^2(x|+-\rangle + y|0+\rangle) = \hbar^2\left\{\frac{7}{4}x|+-\rangle + \sqrt{2}x|0+\rangle + \frac{11}{4}y|0+\rangle + \sqrt{2}y|+-\rangle\right\}$$

$$= \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2\,(x|+-\rangle + y|0+\rangle)$$

となっていると予想する。

これを解くと、

$$\frac{7}{4}x + \sqrt{2}y = \frac{15}{4}x,\quad \sqrt{2}x + \frac{11}{4}y = \frac{15}{4}y$$

$\Longleftrightarrow\ 2x = \sqrt{2}y \ \Longleftrightarrow\ y = \sqrt{2}x$ 　　$x^2+y^2=1$ で規格化すると

$$x = \sqrt{\frac{1}{3}},\quad y = \sqrt{\frac{2}{3}}.$$

$$\therefore\ S^2\left(\sqrt{\frac{1}{3}}|+-\rangle + \sqrt{\frac{2}{3}}|0+\rangle\right) = \frac{3}{2}\left(\frac{3}{2}+1\right)\hbar^2\left(\sqrt{\frac{1}{3}}|+-\rangle + \sqrt{\frac{2}{3}}|0+\rangle\right)$$

(ウラへ)

さらに、これは

$$S^2\left(\sqrt{\tfrac{1}{3}}|-+\rangle+\sqrt{\tfrac{2}{3}}|0-\rangle\right)=\hbar^2\left\{\frac{7}{4\sqrt{3}}|-+\rangle+\sqrt{\tfrac{2}{3}}|0-\rangle+\frac{11}{4}\sqrt{\tfrac{2}{3}}|0-\rangle+\frac{2}{\sqrt{3}}|-+\rangle\right\}$$

$$=\hbar^2\left(\frac{15}{4}\sqrt{\tfrac{1}{3}}|-+\rangle+\frac{15}{4}\sqrt{\tfrac{2}{3}}|0-\rangle\right)$$

$$=\frac{15}{4}\hbar^2\left(\sqrt{\tfrac{1}{3}}|-+\rangle+\sqrt{\tfrac{2}{3}}|0-\rangle\right)$$

についても成り立っていることがわかる。

同様にして

$$S^2(x'|+-\rangle-y'|0+\rangle)=\hbar^2\left\{\frac{7}{4}x'|+-\rangle+\sqrt{2}x'|0+\rangle-\frac{11}{4}y'|0+\rangle-\sqrt{2}y'|+-\rangle\right\}$$

$$=\hbar^2\left(\frac{7}{4}x'|+-\rangle-\sqrt{2}y'|+-\rangle+\sqrt{2}x'|0+\rangle-\frac{11}{4}y'|0+\rangle\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}+1\right)(x'|+-\rangle-y'|0+\rangle)$$

を解いて、$x'=\sqrt{\tfrac{2}{3}}$, $y'=\sqrt{\tfrac{1}{3}}$

$$\therefore\ S^2\left(\sqrt{\tfrac{2}{3}}|+-\rangle-\sqrt{\tfrac{1}{3}}|0+\rangle\right)=\frac{3}{4}\hbar^2\left(\sqrt{\tfrac{2}{3}}|+-\rangle-\sqrt{\tfrac{1}{3}}|0+\rangle\right)$$

これは、

$S^2\left(\sqrt{\tfrac{2}{3}}|-+\rangle-\sqrt{\tfrac{1}{3}}|0-\rangle\right)=\frac{3}{4}\hbar^2\left(\sqrt{\tfrac{2}{3}}|-+\rangle-\sqrt{\tfrac{1}{3}}|0-\rangle\right)$ も満たしている。

まとめると、

$$S^2\xrightarrow[\text{Total}-S]{}\hbar^2\begin{bmatrix}3/2&&&&&\\&3/2&&&0&\\&&3/2&&&\\&&&3/2&&\\&0&&&1/2&\\&&&&&1/2\end{bmatrix}\begin{bmatrix}|++\rangle\\\sqrt{\tfrac{1}{3}}|+-\rangle+\sqrt{\tfrac{2}{3}}|0+\rangle\\\sqrt{\tfrac{2}{3}}|0-\rangle+\sqrt{\tfrac{1}{3}}|-+\rangle\\|--\rangle\\\sqrt{\tfrac{2}{3}}|+-\rangle-\sqrt{\tfrac{1}{3}}|0+\rangle\\-\sqrt{\tfrac{1}{3}}|0-\rangle+\sqrt{\tfrac{2}{3}}|-+\rangle\end{bmatrix}\quad \text{for } \frac{3}{2}\oplus\frac{1}{2}$$

従って、Colbesh-Gordon 行列は、

$$\begin{pmatrix}|\frac{3}{2},\frac{3}{2}\rangle\\|\frac{3}{2},\frac{1}{2}\rangle\\|\frac{3}{2},-\frac{1}{2}\rangle\\|\frac{3}{2},-\frac{3}{2}\rangle\\|\frac{1}{2},\frac{1}{2}\rangle\\|\frac{1}{2},-\frac{1}{2}\rangle\end{pmatrix}=\begin{bmatrix}1&0&0&0&0&0\\0&\sqrt{\tfrac{1}{3}}&\sqrt{\tfrac{2}{3}}&0&0&0\\0&0&0&\sqrt{\tfrac{2}{3}}&\sqrt{\tfrac{1}{3}}&0\\0&0&0&0&0&1\\0&\sqrt{\tfrac{2}{3}}&-\sqrt{\tfrac{1}{3}}&0&0&0\\0&0&0&-\sqrt{\tfrac{1}{3}}&\sqrt{\tfrac{2}{3}}&0\end{bmatrix}\begin{bmatrix}|++\rangle\\|+-\rangle\\|0+\rangle\\|0-\rangle\\|-+\rangle\\|--\rangle\end{bmatrix}$$

P.453　(17.1.16)、(17.1.17)の導出

$$H^0|n^2\rangle + H'|n'\rangle = E_n^0|n^2\rangle + E_n^1|n'\rangle + E_n^2|n^0\rangle$$

$$\langle n^0|H^0|n^2\rangle + \langle n^0|H'|n'\rangle = \langle n^0|E_n^0|n^2\rangle + \langle n^0|E_n^1|n'\rangle + \overbrace{\langle n^0|E_0^2|n^0\rangle}^{E_n^2}$$

$$\Leftrightarrow E_n^2 = \underbrace{\langle n^0|H^0|n^2\rangle}_{=E_n^0\cancel{\langle n^0|n^2\rangle}} + \langle n^0|H'|n'\rangle - E_n^0\cancel{\langle n^0|n^2\rangle} - E_n^1\langle n^0|n'\rangle$$

$$\left(\because\ \langle n^0|H^0 = \langle n^0|E_n^0\right)$$

$$= \langle n^0|H'|n'\rangle - E_n^1\underbrace{\langle n^0|n'\rangle}_{=\langle n^0|n'_{\parallel}\rangle = i\alpha = 1-e^{i\alpha}\equiv 0}$$

$$= \langle n^0|H'|n'\rangle$$

ところで、 $|n'\rangle = \sum_m{}' \dfrac{|m^0\rangle\langle m^0|H'|n^0\rangle}{E_n^0 - E_m^0}$　(17.1.13)

$$\therefore E_n^2 = \langle n^0|H'\sum_m{}'\frac{|m^0\rangle\langle m^0|H'|n^0\rangle}{E_n^0 - E_m^0}$$

$$= \sum_m{}'\frac{\langle n^0|H'|m^0\rangle\langle m^0|H'|n^0\rangle}{E_n^0 - E_m^0} = \sum_m{}'\frac{|\langle n^0|H'|m^0\rangle|^2}{E_n^0 - E_m^0} \quad (\leftarrow 17.1.16)$$

$$\left(\because\ \langle m^0|H'|n^0\rangle = \langle n^0|H'|m^0\rangle^*\right)$$

(17.2.6)の導出

$$E_n^2 = \sum_m{}'\frac{|\langle m^0|-qf\sqrt{\frac{\hbar}{2m\omega}}(a+a^+)|n^0\rangle|^2}{E_n^0 - E_m^0}$$

マトリックスエレメントは $m^0 = n^0 \pm 1$ のみが非ゼロで、あとは0。

$$\therefore E_n^2 = \left[\frac{|\langle m^0|-qf\sqrt{\frac{\hbar}{2m\omega}}(a+a^+)|n^0\rangle|^2}{E_n^0 - E_m^0}\right]_{m^0=n^0+1} + \left[\frac{|\langle m^0|-qf\sqrt{\frac{\hbar}{2m\omega}}(a+a^+)|n^0\rangle}{E_n^0 - E_m^0}\right]_{m^0=n^0-1}$$

$$= \frac{|-qf\sqrt{\frac{\hbar}{2m\omega}}\langle n^0+1|a+a^+|n^0\rangle|^2}{E_n^0 - E_{n+1}^0} + \frac{|-qf\sqrt{\frac{\hbar}{2m\omega}}\langle n^0-1|a+a^+|n^0\rangle|^2}{E_n - E_{n-1}^0}$$

$$= q^2f^2\frac{\hbar}{2m\omega}\left(\frac{n+1}{-\hbar\omega} + \frac{n}{\hbar\omega}\right) = -\frac{q^2f^2}{2m\omega^2}$$

Exercise 17.2.1

$$H' = \lambda x^4$$

$$E_n^1 = \langle n^0 | H' | n^0 \rangle = -\lambda \langle n^0 | X^4 | n^0 \rangle = -\lambda^2 \left(\frac{\hbar}{2m\omega}\right)^2 \langle n^0 | (a+a^+)^4 | n^0 \rangle$$

$$= -\lambda \left(\frac{\hbar}{2m\omega}\right)^2 \langle n^0 | \cancel{a^4} + \cancel{a^3 a^+} + \cancel{a^2 a^+ a} + a^2 a^{+2}$$
$$+ \cancel{a a^+ a^2} + a a^+ a a^+ + a a^{+2} a + \cancel{a a^{+3}}$$
$$+ \cancel{a^+ a^3} + a^+ a^2 a^+ + a^+ a a^+ a + \cancel{a^+ a a^{+2}}$$
$$+ a^{+2} a^2 + \cancel{a^{+2} a a^+} + \cancel{a^{+3} a} + \cancel{a^{+4}} \, | n^0 \rangle$$

$$= -\lambda \left(\frac{\hbar}{2m\omega}\right)^2 \langle n^0 | a^2 a^{+2} + a a^+ a a^+ + a a^{+2} a + a^+ a^2 a$$
$$+ a^+ a a^+ a + a^{+2} a^2 | n^0 \rangle$$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$ を使って

$$a^2 a^{+2} |n\rangle = a^2 a^+ \sqrt{n+1}\,|n+1\rangle = a^2 \sqrt{n+1}\sqrt{n+2}\,|n+2\rangle$$
$$= a\sqrt{n+1}\,(n+2)\,|n+1\rangle = (n+1)(n+2)\,|n\rangle$$

$$a a^+ a a^+ |n\rangle = a a^+ a \sqrt{n+1}\,|n+1\rangle = a a^+ (n+1)\,|n\rangle = (n+1)^2 |n\rangle$$

$$a a^{+2} a |n\rangle = a a^{+2} \sqrt{n}\,|n-1\rangle = a a^+ n |n\rangle = (n+1) n\,|n\rangle$$

$$a^+ a^2 a^+ |n\rangle = a^+ a^2 \sqrt{n+1}\,|n+1\rangle = a^+ a (n+1)\,|n\rangle = n(n+1)\,|n\rangle$$

$$a^+ a a^+ a |n\rangle = a^+ a a^+ \sqrt{n}\,|n-1\rangle = a^+ a n |n\rangle = n^2 |n\rangle$$

$$a^{+2} a^2 |n\rangle = a^{+2} a \sqrt{n}\,|n-1\rangle = a^+ \sqrt{n}\sqrt{n-1}\,|n-2\rangle$$
$$= a^+ \sqrt{n}\,(n-1)\,|n-1\rangle = n(n-1)\,|n\rangle$$

$$\therefore E_n^1 = -\lambda \left(\frac{\hbar}{2m\omega}\right)^2 \left\{ (n+1)(n+2) + (n+1)^2 + 2n(n+1) + n^2 + n(n-1) \right\}$$
$$= -\frac{3\lambda\hbar^2}{4m\omega^2}(2n^2 + 2n + 1)$$

For oscillator, $\Delta E_n^0 \equiv E_n^0 - E_{n-1}^0 = \hbar\omega$

$$\frac{E_n^1}{\Delta E_n^0} = -\frac{3\lambda\hbar}{4m\omega}(2n^2+2n+1)$$ At some large n, $E_n^1 > \Delta E_n^0$